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Quaternion Polynomials
and
Rational Rotation–Minimizing Frame Curves

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Abstract

The study of frames associated with a spatial curve is an active research field. More precisely, the attention has focused on *adapted* frames, whose one of the vectors of the orthonormal triple coincides with the curve unit tangent at each point. Among the adapted frames, the *rotation–minimizing frame* (RMF) is useful for applications such as swept surfaces constructions and animation, since it executes the least possible frame rotation along the curve. Rational representation of RMF is desirable in computer aided design applications. Hence, recent studies focus on *rational rotation–minimizing frames* (RRMFs) and moreover on the identification, characterization and construction of curves with RRMFs. The curves with rational adapted frames are necessarily curves with rational unit tangent vector, known as *Pythagorean–hodograph* (PH) curves. The *quaternion* and *Hopf map* forms are two alternative models for the representation of PH curves. Rotation–minimizing frames are not in general rational, even for PH curves. The Euler–Rodrigues frame (ERF), which is always defined on a PH curve and is rational by its construction, is a good reference for identifying rational RMFs on PH curves. The ERF is not in general an RMF. The simplest non–planar curves for which the ERF can be an RMF are of degree 7.

In the present thesis we give a characterization of degree 7 spatial PH curves with rotation–minimizing Euler–Rodrigues frame using both the quaternion and Hopf map forms. Further, we deal with the quintic PH curves and investigate conditions under which a quintic is an RRMF curve of particular type in terms of the quaternion and Hopf map form. Since quaternion polynomials generate the hodographs of the PH curves, we also focus on the study of the quaternion algebra and present results pertinent in the study of RRMF curves. An algorithm for the roots of quadratic quaternion polynomials is presented and used to analyze the root structure of polynomials that generate quintic curves with RRMFs. Finally, we

prove that PH curves with *non–primitive* hodographs are those whose its associated quaternion polynomial has a right complex factor and we see that such curves are regular curves. Further, we give necessary and sufficient conditions for a quaternion polynomial to have a right complex factor. Throughout this study we also identify and characterize some particular types of PH curves which are generated by others of lower degree.

ΠΕΡΙΛΗΨΗ

Η μελέτη των συστημάτων συντεταγμένων ή πλαισίων (frames) τα οποία ορίζονται πάνω σε μία χωροκαμπύλη αποτελεί ένα πολύ ενδιαφέρον επιστημονικό πεδίο έρευνας. Πιο συγκεκριμένα, ενδιαφερόμαστε για εκείνα τα πλαίσια τα οποία είναι ορθοκανονικά και στα οποία το ένα από τρία διανύσματα συμπίπτει με το εφαπτόμενο διάνυσμα της καμπύλης, σε κάθε σημείο της. Τέτοια πλαίσια τα ονομάζουμε προσαρμοσμένα πλαίσια και μεταξύ αυτών ενδιαφερόμαστε ιδιαιτέρως για τα πλαίσια ελάχιστης περιστροφής (RMF) τα οποία έχουν εξαιρετικά σημαντικές εφαρμογές, αφού εκτελούν την ελάχιστη περιστροφή κατα μήκος της καμπύλης. Ρητές αναπαραστάσεις των RMF ειναι επιθυμητές στις εφαρμογές, και έτσι οι τελευταίες έρευνες έχουν επικεντρωθεί στην μελέτη τέτοιων πλαισίων- που καλούνται RRMF - και κυρίως στον προσδιορισμό, χαρακτηρισμό και κατασκευή καμπυλών στα οποία μπορούν να οριστούν RRMF πλαίσια. Αυτές οι καμπύλες πρέπει απαραιτήτως να είναι καμπύλες με ρητό εφαπτομενικό μοναδιαίο διάνυσμα, οι οποίες είναι γνωστές ως καμπύλες με πυθαγόρεια οδογραφήματα (PH καμπύλες). Χρησιμοποιώντας τα πολυώνυμα με συντελεστές τετραδικούς αριθμούς (quaternions) ή εναλλακτικά την απεικόνιση Hopf μπορούμε να αναπαραστήσουμε τις PH καμπύλες. Όμως, ακόμα και στις PH καμπύλες ένα RMF, δεν είναι πάντοτε ρητό. Το Euler–Rodrigues πλαίσιο (ERF) που ορίζεται σε κάθε PH καμπύλη και είναι εκ κατασκευής ρητό, αποτελεί μία καλή αναφορά για τον προσδιορισμό RRMF στις PH καμπύλες. Το ERF δεν είναι εν γένει RMF. Οι μικρότερου βαθμού μη επίπεδες καμπύλες για τις οποίες το ERF μπορεί να είναι RMF είναι οι καμπύλες 7ου βαθμού.

Στην παρούσα διατριβή δίνουμε ένα χαρακτηρισμό των PH καμπυλών 7ου βαθμού στις οποίες το ERF είναι ένα RMF, χρησιμοποιώντας και τις δύο ισοδύναμες μορφές αναπαράστασης των. Επιπλέον, ασχολούμαστε με τις PH καμπύλες 5ου βαθμού και ερευνούμε τις συνθήκες κάτω από τις οποίες μία τέτοια καμπύλη είναι RRMF συγκεκριμένης κατηγορίας. Επίσης, μελετάμε τα πολυώνυμα με συντελεστές τετραδικούς αριθμούς, αφού μέσω αυτών των πολυωνύμων εκφράζουμε το οδογράφημα των PH καμπυλών και παρουσιάζουμε σχετικά αποτελέσματα που μας βοηθούν στην μελέτη των RRMF καμπυλών. Ακόμα παρουσιάζουμε έναν αλγόριθμο που υπολογίζει τις ρίζες των πολυωνύμων 2ου βαθμού με συντελεστές τετραδικούς αριθμούς και ο οποίος χρησιμοποιείται στην μελέτη των RRMF καμπυλών 5ου βαθμού. Τέλος, αποδεικνύουμε ότι οι PH καμπύλες με μη πρωτογενή οδογραφήματα είναι αυτές στις οποίες το αντίστοιχο πολυώνυμο με συντελεστές τετραδικούς αριθμούς μέσω του οποίου εκφράζεται, έχει μιγαδική ρίζα και παρατηρούμε ότι αυτές οι καμπύλες είναι ομαλές καμπύλες. Επιπλέον, δίνουμε μία ικανή και αναγκαία συνθήκη για ένα

τέτοιο πολυώνυμο να έχει μία τουλάχιστον μιγαδική ρίζα. Επιπλέον, δια μέσου αυτής της μελέτης προσδιορίζουμε και χαρακτηρίζουμε κάποιες συγκεκριμένες κατηγορίες καμπυλών που "παράγονται" από άλλες μικρότερου βαθμού.

Introduction

The *hodograph* of a smooth parametric curve $\mathbf{r}(t) = (x(t), y(t), z(t))$ in $\mathbb{R}^3$ is the locus described by its derivative $\mathbf{r}'(t) = (x'(t), y'(t), z'(t))$. The geometric properties of hodographs may offer useful information for curve design and analysis problems. In this thesis, we shall be concerned with polynomial curves $\mathbf{r}(t)$ whose hodograph satisfy the Pythagorean condition

$$x'^2(t) + y'^2(t) + z'^2(t) = \sigma^2(t),$$

for some real polynomial $\sigma(t)$. These curves are called *Pythagorean hodograph (PH)* curves and were first introduced by Farouki and Sakkalis in 1990 [30], as a special class of polynomial curves with significant properties in Computational Geometry. By their definition, PH curves are distinguished by the property that their "parametric speed", i.e. the rate of change of the arc length s with respect to the parameter t, is just a polynomial, rather than the square root of polynomial, in t. This feature dowers PH curves with interesting computational advantages over "ordinary" parametric curves. Some of them are cited below:

1. The polynomial *arc length* function

   $$s(t) = \int_0^t |\mathbf{r}'(u)| \mathrm{d}u$$

   admits exact computation [17].

2. The *offsets* $\mathbf{r}_d(t) = \mathbf{r}(t) + d\,\mathbf{n}(t)$ to any planar PH curve, -i.e., loci of points which have a constant distance from the curve- and the *pipe* or *canal surfaces* that have a given spatial PH curve as a spine, admit an exact rational parametrization [31, 30, 28].

3. The *energy integral*

   $$E = \int_0^S k^2 \mathrm{d}s$$

i.e., the integral of the square of the curvature, has exact closed-form evaluation.

4. Rational unit tangent, curvature, *rational adapted frames,* etc, can be exactly computed.

5. *Rotation-minimizing frames*, which eliminate the "unnecessary" rotation of the Frenet frame in the curve normal plane, may be exactly derived. For spatial PH curves even though these involve logarithmic terms [18], otherwise rational approximations are available [25].

Since in a variety of applications, such as robotics, animation, computer graphics, motion control and swept surfaces constructions is desirable for the shapes to be expressed by rational representations, PH curves are suited for applications in computer aided design and manufacturing. More precisely, in many of the above applications a basic problem appears to be the need to describe the orientation of a rigid body that moves along a given trajectory. This can be accomplished by invoking an orthonormal frame $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ embedded in the body to specify its orientation. Frames that incorporate the unit tangent as one component are known as *adapted* and an additional desirable property for their components is the rational dependence of the curve parameter. The most familiar adapted frame is the Frenet frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ [1] but it is often unsuitable for practical applications since it is not defined at inflection points and moreover incurs an "unnecessary" rotation of the normal plane vectors $\mathbf{n}, \mathbf{b}$ around $\mathbf{t}$. To address this problem, the construction of *rotation- minimizing adapted frames* (RMF) $(\mathbf{t}, \mathbf{f}_1, \mathbf{f}_2)$ has been the focus of recent research. The advantages of RMFs for construction of swept surfaces were first studied by Klock [58] who characterized them as solutions of first order differential equations. RMFs have the property of minimum twist and makes them useful in several topics such as motion design and control in computer animations, robotics, and tool path planning in CAD [80]. Numerical methods are often used [25, 58, 74, 59, 80, 81] to approximate RMFs, since in general the unit vectors $\mathbf{t}, \mathbf{f}_1, \mathbf{f}_2$ do not admit a rational dependence of the curve parameter even if $\mathbf{r}(t)$ is a polynomial or rational curve.

In view of the above, it is clear that for CAD applications rational representations are preferable in order to have exact computations. On the other hand, RMFs are desirable since not only identify the orientation of a moving body along a path but furthermore they possess the property of minimum variation. Therefore, an important "requirement" for computer aided design applications is that an RMF be rationally dependent. This is the reason why there has been recently considerable interest [21, 24, 34, 32] in a special subset of the PH curves, known as the *rational rotation-minimizing frame (RRMF) curves.* These curves possess rational orthonormal adapted frames $(\mathbf{t}, \mathbf{u}, \mathbf{v})$, where $\mathbf{t} = \mathbf{r}'(t)/|\mathbf{r}'(t)|$ is the curve tangent and $\mathbf{u}, \mathbf{v}$ span the curve normal plane at each point, with a frame angular velocity ω that satisfies $\omega \cdot \mathbf{t} \equiv \mathbf{0}$. All RRMF curves must be necessarily PH curves since only PH curves have rational unit tangents and they may be characterized by an algebraic conditions on the coefficients of the *quaternion* and *Hopf map* representations [21, 23]. But the PH property alone does not ensure the existence of a rational RMF. Although any spatial PH curve supports an exact computation of RMF via integration of a rational function [18], these RMFs do not have a rational dependence on the curve parameter because, in general, rational functions do not have rational integrals. Thus, construction of curves that possess an RRMF is a difficult task since non-linear constraints are involved and research is focusing on identifying constraints on the coefficients of PH curves which are necessary and sufficient for rational RMF. More results on construction, applications and rational approximations of RMF curves were investigated in [24, 25, 21, 32, 55] and the citations therein.

In [7], Choi and Han introduced a special adapted frame, called ERF, defined for any spatial PH curve, which is rational by construction and it has non-singular behavior at inflection points. Unfortunately, the ERF is not in general an RMF. Also in [7] conditions under which the ERF of a PH curve can be an RMF were studied, and it was shown that:

- for PH cubics ERF and Frenet frame are the same
- the PH quintics which have rotation minimizing ERF are planar curves
- spatial PH curves for which the ERF can be RMF are of degree 7 at least and such curves were shown to depend on 16 real parameters

Subsequently, Han [45] using the ERF as a key step for identifying RRMFs, formulated an algebraic criterion characterizing RRMF curves of any (odd) degree and furthermore proved that RRMF cubics are either planar or PH curves with non–primitive hodographs. The simplest true spatial curves with RRMFs are quintics and there were investigated in [34, 23, 21, 33]. More precisely, in [23] was first presented the existence of non-degenerate RRMF quintics and characterized in terms of coefficient constraints in the Hopf map representation. Later on, in [21] these conditions were replaced by a simpler and more concise condition in terms of the coefficient of both quaternion and Hopf map representations. Finally, general degree PH curves were treated in [32] and Han's criterion was studied in case of PH curves of arbitrary degree.

The focus of this thesis is on the identification and characterization of some remarkable types of quintics and degree 7 PH curves with associated rational frames of special interest. The structure of the thesis consists of the following parts: the part which includes Chapters 2 through 4 and it presents the basic theory of quaternion polynomials together with some results pertinent to this research, and the other one includes Chapters 5 through 7 and presents the conditions which characterize some particular types of RRMF curves.

We now briefly summarize the contents of each chapter. Chapter 2 is devoted to the presentation of basic facts about quaternions and quaternion polynomials. In particular, some classical and recent results on the roots and the factorization of quaternion polynomials are given. In Chapter 3 we study the complex roots of quaternion polynomials. We give necessary and sufficient conditions in terms of Bézout matrices for a quaternion to have a complex, a spherical and a complex isolated root. Furthermore, we give a bound for the size of the roots. Chapter 4 deals with quadratic quaternion polynomials. We recall some known results on the factorization of a quadratic quaternion polynomial and give conditions, in terms of real variables, in order for a quadratic equation to have a special kind and specific multiplicity of roots. In addition, we present a new algorithm for finding the roots of a quadratic quaternion equation which is used to analyze the root structure of the quaternion polynomial that generates quintic RRMF curves. Chapter 5 summarizes some basic facts of the theory of adapted frames and of PH curves. Chapter 6 is devoted to the study of PH curves of degree 5 and 7 and gives necessary and sufficient conditions in terms of their associated quaternion polynomial so that the curves are of a certain type. Finally, in Chapter 7, we consider the problem of characterizing the non–primitive hodographs generated by a primitive quaternion polynomial. More precisely, we prove that a hodograph is non–primitive if and only if the associated quaternion polynomial has a right complex factor. Although, in general, we "avoid" having non–primitive hodographs, we are interested for these where the PH curves are regular. Through the study of non–primitive hodographs, we can see that there are RRMF curves which are "generated" by others. Consequently, we give necessary and sufficient conditions for an RRMF curve to be generated by another of lower degree in terms of its associated quaternion polynomial and we study some of their geometrical properties. Moreover, we apply the previous results in order to identify and characterize these specific types of quintics and degree 7 PH curves. The latter, this leads to the construction of an RRMF curve by another of lower degree.

Chapter 1

Quaternions and Quaternion Polynomials

The purpose of this chapter is to recall some basic facts about quaternions and quaternion polynomials which we shall use for the presentation of our results. First, Section 1.1 introduces the concept of *quaternion* numbers with some of their properties and Section 1.2 briefly reviews some basic facts about quaternion polynomials. Then, Sections 1.3 and 1.4 are devoted to root finding as well as the factorization of these polynomials. In order to become more familiar with these concepts, some examples are presented as well.

1.1 The skew field of quaternions

Recall that a ring $\mathbf{R}$ is called a *division ring* if every non–zero element of $\mathbf{R}$ has a two–sided inverse. A division ring may be commutative, in which case it is a *field*, or non–commutative, in which case it is a *skew field*. We are concerned here with the most familiar example of a non–commutative division ring, namely, the ring of Hamilton's *quaternions*.

Quaternions were first introduced by W. R. Hamilton in an attempt to extend the set of complex numbers to higher dimensions. They differ from the complex numbers since they involve three imaginary units, rather than just one. Moreover, these three imaginary units are non–commutative, and consequently the quaternion product is non–commutative. Apart from their theoretical importance as the first example of a non–commutative algebra, the ability of quaternions to describe rotations in $\mathbb{R}^3$ makes them extremely useful in applications such computer graphics, robotics, computer–aided design, manufacturing and animation.

We begin by reviewing some basic definitions and properties of the quaternions.

Definition 1.1 A *quaternion* is an expression of the form

$$\mathcal{Q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k},$$

where $q_0, q_1, q_2, q_3 \in \mathbb{R}$ and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ satisfy the multiplication rules

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}. \tag{1.1}$$

We use calligraphic characters $\mathcal{A}, \mathcal{B}, \ldots$ to denote quaternions, bold characters $\mathbf{a}, \mathbf{b}, \ldots$ for complex numbers (or vectors in $\mathbb{R}^3$ — the meaning will be clear from the context), and italic characters $a, b, c, \ldots$ for real numbers. Also, the quaternion element $\mathbf{i}$ will always be identified with the imaginary unit i.

We denote by $\mathbb{H}$ the set of all quaternions. Let

$$\mathcal{Q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \quad \text{and} \quad \mathcal{P} = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$$

be two elements of $\mathbb{H}$. The addition and the multiplication in $\mathbb{H}$ are defined as follows:

$$\mathcal{Q} + \mathcal{P} = q_0 + p_0 + (q_1 + p_1)\mathbf{i} + (q_2 + p_2)\mathbf{j} + (q_3 + p_3)\mathbf{k}$$

and

$$\begin{aligned} \mathcal{QP} &= (q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3) + (q_0p_1 + q_1p_0 + q_2p_3 - q_3p_2)\,\mathbf{i} \\ &+ (q_0p_2 + q_2p_0 + q_3p_1 - q_1p_3)\,\mathbf{j} + (q_0p_3 + q_3p_0 + q_1p_2 - q_2p_1)\,\mathbf{k}. \end{aligned}$$

The set $\mathbb{H}$ with these two operations is a division ring.

Every element of $\mathbb{H}\backslash\mathbb{R}$ is called a *pure* quaternion. For a given quaternion $\mathcal{Q}$, we call $q_0 = \mathrm{scal}(\mathcal{Q})$ the *scalar* (or *real*) part of $\mathcal{Q}$, and $q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} = \mathrm{vect}(\mathcal{Q})$ the *vector* (or *imaginary*) part of $\mathcal{Q}$. Clearly, the set of real numbers $\mathbb{R}$ and the set of complex numbers $\mathbb{C}$ are (commutative) subrings of $\mathbb{H}$, corresponding to the sets of quaternions with $q_1 = q_2 = q_3 = 0$ and $q_2 = q_3 = 0$ respectively. The quaternion ring defines a 4–dimensional real vector space, with basis elements $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ when addition is performed component–wise.

By swapping q_0, q_1, q_2, q_3 and p_0, p_1, p_2, p_3 in the above formula, one can verify that in general $\mathcal{QP}$ and $\mathcal{PQ}$ have identical scalar parts, but different vector parts.

The *conjugate* $\mathcal{Q}^*$ of $\mathcal{Q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is defined by

$$\mathcal{Q}^* = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k},$$

and it is easily seen that

$$\mathcal{Q}\mathcal{Q}^* = \mathcal{Q}^*\mathcal{Q} = q_0^2 + q_1^2 + q_2^2 + q_3^2.$$

We call *norm* of $\mathcal{Q}$ the real number

$$|\mathcal{Q}| = \sqrt{\mathcal{Q}\mathcal{Q}^*} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

One can easily verify that

$$|\mathcal{Q}|^2 = (\mathrm{scal}(\mathcal{Q}))^2 + |\mathrm{vect}(\mathcal{Q})|^2.$$

Further, for every $\mathcal{Q}, \mathcal{P} \in \mathbb{H}$ we have

$$|\mathcal{Q}\mathcal{P}| = |\mathcal{Q}||\mathcal{P}| \quad \text{and} \quad |\mathcal{Q} + \mathcal{P}| \leq |\mathcal{Q}| + |\mathcal{P}|.$$

A quaternion $\mathcal{U}$ with $|\mathcal{U}| = 1$ is called a *unit* quaternion. Any non–zero quaternion $\mathcal{Q}$ has a multiplicative inverse, i.e. an element, denoted by $\mathcal{Q}^{-1}$, such that

$$\mathcal{Q}^{-1}\mathcal{Q} = \mathcal{Q}\mathcal{Q}^{-1} = 1.$$

The element $\mathcal{Q}^{-1}$ is unique and is given by

$$\mathcal{Q}^{-1} = \frac{\mathcal{Q}^*}{|\mathcal{Q}|^2}.$$

A quaternion $\mathcal{Q}$ may be written as $\mathcal{Q} = (q, \mathbf{q})$ where $q = \mathrm{scal}(\mathcal{Q})$ and $\mathbf{q} = \mathrm{vect}(\mathcal{Q})$. Correspondingly, we can express the *conjugate* and the *norm* of $\mathcal{Q}$ by

$$\mathcal{Q}^* = (q, -\mathbf{q}), \qquad |\,\mathcal{Q}\,| = \sqrt{q^2 + |\mathbf{q}|^2}\,.$$

The sum and product of given quaternions $\mathcal{A} = (a, \mathbf{a})$ and $\mathcal{B} = (b, \mathbf{b})$ may be compactly expressed [70] as

$$\mathcal{A} + \mathcal{B} = (a + b\,,\, \mathbf{a} + \mathbf{b})\,, \qquad \mathcal{A}\mathcal{B} = (ab - \mathbf{a} \cdot \mathbf{b}\,,\, a\,\mathbf{b} + b\,\mathbf{a} + \mathbf{a} \times \mathbf{b})\,,$$

where $\cdot$ and $\times$ denote the usual vector dot and cross products in $\mathbb{R}^3$.

For brevity, we shall simply write q and $\mathbf{q}$ for pure scalar and pure vector quaternions of the form $(q, \mathbf{0})$ and $(0, \mathbf{q})$.

We now introduce the notion of *similarity* for any two quaternions $\mathcal{Q}$ and $\mathcal{P}$.

Definition 1.2 Two quaternions $\mathcal{Q}$ and $\mathcal{P}$ are said to be *similar* and we write $\mathcal{Q} \sim \mathcal{P}$, if there exists a non–zero quaternion $\mathcal{S}$ such that

$$\mathcal{P} = \mathcal{S}\mathcal{Q}\mathcal{S}^{-1}.$$

Equivalently, $\mathcal{Q} \sim \mathcal{P}$ if there exists a non–zero quaternion $\mathcal{S}$, such that $\mathcal{P}\mathcal{S} = \mathcal{S}\mathcal{Q}$.

It is easily verified that $\sim$ is an equivalence relation. For every $\mathcal{Q} \in \mathbb{H}$ the equivalence class of $\mathcal{Q}$ is the set

$$[\,\mathcal{Q}\,] := \{\mathcal{P} \in \mathbb{H} \,:\, \mathcal{P} \sim \mathcal{Q}\}.$$

The following proposition allows us to easily check whether or not two quaternions are similar.

Proposition 1.1. *Let $\mathcal{Q}$ and $\mathcal{P}$ be quaternions. Then $\mathcal{Q} \sim \mathcal{P}$ if and only if*

$$\mathrm{scal}(\mathcal{Q}) = \mathrm{scal}(\mathcal{P}) \qquad \textit{and} \qquad |\mathrm{vect}(\mathcal{Q})| = |\mathrm{vect}(\mathcal{P})|.$$

Proof: See [51]. ∎

Corollary 1.1. *For a (pure) quaternion $\mathcal{Q} \in \mathbb{H} \setminus \mathbb{R}$, the equivalence class $[\,\mathcal{Q}\,]$ always has infinitely many elements.*

Similar quaternions can directly be identified using Proposition 1.1. The complex number $\mathbf{z} = \mathrm{scal}(\mathcal{Q}) + |\mathrm{vect}(\mathcal{Q})|\,\mathrm{i}$ is the only complex number similar to $\mathcal{Q}$ with a positive imaginary part, and is called the *complex similar* of $\mathcal{Q}$. Note that, for a given quaternion $\mathcal{Q}$, there are at most two complex numbers in its equivalence class $[\,\mathcal{Q}\,]$ — the complex numbers $\mathbf{z}$ and $\mathbf{z}^*$.

Now suppose $\mathcal{Q} \in \mathbb{R}$. Then $[\,\mathcal{Q}\,] = \{\mathcal{Q}\}$, which means that the equivalence class contains only the single element $\mathcal{Q}$. If $\mathcal{Q} \notin \mathbb{R}$, then $[\,\mathcal{Q}\,]$ always contains infinitely–many elements, and can be expressed as

$$[\,\mathcal{Q}\,] = \{\, \mathcal{P} \in \mathbb{H} \,:\, \mathrm{scal}(\mathcal{P}) = \mathrm{scal}(\mathcal{Q}) \ \text{ and } \ |\mathrm{vect}(\mathcal{P})| = |\mathrm{vect}(\mathcal{Q})| \,\}.$$

Obviously, we have $\mathcal{Q}^* \in [\,\mathcal{Q}\,]$ for all $\mathcal{Q} \in \mathbb{H}$.

Example 1.1.1 Let $\mathcal{Q} = 2 - \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$. Then the equivalence class of $\mathcal{Q}$ is

$$[\,\mathcal{Q}\,] = \{\,\mathcal{P} \in \mathbb{H} \,:\, \mathrm{scal}(\mathcal{P}) = 2 \text{ and } |\mathrm{vect}(\mathcal{P})| = \sqrt{14}\,\}.$$

The only complex numbers in $[\,\mathcal{Q}\,]$ are $\mathbf{c} = 2 + \sqrt{14}\,\mathrm{i}$ and $\mathbf{c}^* = 2 - \sqrt{14}\,\mathrm{i}$.

Remark 1.1. If we identify the quaternion $\mathcal{Q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ as a point in $\mathbb{R}^4$ and denote it as $\mathcal{Q} = (q_0, q_1, q_2, q_3)$, then the set $[\,\mathcal{Q}\,]$ of quaternions similar to $\mathcal{Q}$ can be regarded geometrically as a *two–dimensional sphere* S^2 in $\mathbb{R}^4$ with center $(q_0, 0, 0, 0)$ and radius $|\mathrm{vect}(\mathcal{Q})| = \sqrt{q_1^2 + q_2^2 + q_3^2}$. We use this interpretation in the following section.

We list below some important properties of quaternions.

Proposition 1.2. [82] *For any quaternions $\mathcal{Q}$, $\mathcal{P}$ the following properties hold:*

1. $|\mathcal{Q}|^2 = |\mathrm{scal}(\mathcal{Q})|^2 - |\mathrm{vect}(\mathcal{Q})|^2 + 2\,|\mathrm{scal}(\mathcal{Q})|\,|\mathrm{vect}(\mathcal{Q})|$.

2. $|\mathcal{Q}|^2 + |\mathcal{P}|^2 = \frac{1}{2}(|\mathcal{Q} + \mathcal{P}|^2 + |\mathcal{Q} - \mathcal{P}|^2)$.

3. $|\mathcal{Q}| = |\mathcal{Q}^*|$.

4. $(\mathcal{Q} + \mathcal{P})^2 \neq \mathcal{Q}^2 + \mathcal{P}^2 + 2\mathcal{Q}\mathcal{P}$ *in general.*

5. $\mathcal{Q} = |\mathcal{Q}|\,\mathcal{U}$, *where $\mathcal{U}$ is a unit quaternion.*

6. *For any $\mathbf{z} \in \mathbb{C}$ we have $\mathbf{j}\,\mathbf{z}\,\mathbf{j}^* = \mathbf{z}^*$ or $\mathbf{j}\,\mathbf{z} = \mathbf{z}^*\,\mathbf{j}$, and $\mathbf{k}\,\mathbf{z}\,\mathbf{k}^* = \mathbf{z}^*$ or $\mathbf{k}\,\mathbf{z} = \mathbf{z}^*\,\mathbf{k}$.*

7. $\mathcal{Q} = \mathcal{Q}^*$ *if and only if* $\mathcal{Q} \in \mathbb{R}$.

8. *Every quaternion $\mathcal{Q}$ can be expressed in the form $\mathcal{Q} = \mathbf{z}_1 + \mathbf{z}_2\,\mathbf{j}$ with $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{C}$.*

9. *Every quaternion $\mathcal{Q}$ can be written in the form $\mathcal{Q} = \mathbf{z}_1 + \mathbf{k}\,\mathbf{z}_2$ with $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{C}$.*

Another useful concept is the *characteristic polynomial* of a quaternion [76, 82]. It can be directly proved that the quaternion $\mathcal{Q} = q_0+q_1\mathbf{i}+q_2\mathbf{j}+q_3\mathbf{k}$ satisfies the quadratic equation

$$\varphi_{\mathcal{Q}}(t) = t^2 - 2\,\mathrm{scal}(\mathcal{Q})\,t + |\mathcal{Q}|^2 = 0 \tag{1.2}$$

with real coefficients. The real quadratic polynomial $\varphi_{\mathcal{Q}}(t)$ is called the characteristic polynomial of $\mathcal{Q}$. We observe that the conjugate $\mathcal{Q}^*$ also satisfies this equation. Moreover, the converse of these statements is also valid — i.e., if $\psi(t) = t^2 + 2\,a\,t + b$ is a real quadratic polynomial, then for any quaternion $\mathcal{P}$ with $\mathrm{scal}(\mathcal{P}) = -a$ and $|\mathcal{P}|^2 = b$, both $\mathcal{P}$ and $\mathcal{P}^*$ satisfy $\psi(t) = 0$. Hence, there are infinitely–many quaternions satisfying $\psi(t) = 0$. These solutions have identical real (scalar) parts, and their imaginary (vector) parts lie on the surface of a two–dimensional sphere in $\mathbb{R}^3$. Thus, a quadratic polynomial over $\mathbb{H}$ may have infinitely many solutions and not at most two, as in the case of a quadratic polynomial over a field. This fact offers a first hint at the more–complicated structure of polynomials in $\mathbb{H}[t]$.

1.2 Quaternion polynomials

In this section we define polynomials with quaternion coefficients, and study some of their key properties which will be used subsequently. As we shall see, the ring of quaternion polynomials is also a non–commutative ring, since the quaternions form a non–commutative division ring. This fact makes the study of quaternion polynomials a fascinating topic that often involves unusual and surprising results.

As usual, let $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, and let $\mathbb{H}$ denote the non–commutative division ring of quaternions. Before proceeding to the definition of quaternion polynomials we note that, due to the non–commutative nature of this ring, there are several ways to define such polynomials. The coefficients can be placed on the right, on the left, or on both sides of the powers of the variable t. We adopt here the practice of defining quaternion polynomials by always writing the powers of t on the right of the quaternion coefficients.

Definition 1.3 A quaternion polynomial $\mathcal{Q}(t)$ in the indeterminate t, is an expression of the form

$$\mathcal{Q}(t) = \mathcal{Q}_n t^n + \cdots + \mathcal{Q}_1 t + \mathcal{Q}_0 = \sum_{i=0}^{n} \mathcal{Q}_i t^i,$$

with $\mathcal{Q}_0, \mathcal{Q}_1, \ldots, \mathcal{Q}_n \in \mathbb{H}$. If $\mathcal{Q}_n \neq 0$ we define the degree of $\mathcal{Q}(t)$, denoted by $\deg \mathcal{Q}(t)$, to be n. When $\mathcal{Q}_n = 1$, we say that $\mathcal{Q}(t)$ is *monic*.

Remark 1.2. $\mathcal{Q}(t)$ can also be expressed in the following equivalent forms:

$$\mathcal{Q}(t) = u(t) + \mathbf{i}\, v(t) + \mathbf{j}\, p(t) + \mathbf{k}\, q(t),$$

or

$$\mathcal{Q}(t) = \boldsymbol{\alpha}(t) + \mathbf{k}\, \boldsymbol{\beta}(t),$$

with

$$\boldsymbol{\alpha}(t) = u(t) + \mathbf{i}\, v(t) \quad \text{and} \quad \boldsymbol{\beta}(t) = q(t) + \mathbf{i}\, p(t) \in \mathbb{C}[t],$$

where $u(t), v(t), p(t), q(t) \in \mathbb{R}[t]$.

We denote by $\mathbb{H}[t]$ the set of quaternion polynomials. The sum and the product of two quaternion polynomials

$$\mathcal{Q}(t) = \sum_{i=0}^{n} \mathcal{Q}_i t^i \quad \text{and} \quad \mathcal{P}(t) = \sum_{j=0}^{m} \mathcal{P}_j t^j$$

are defined as follows:

$$\mathcal{Q}(t) + \mathcal{P}(t) = \sum_{i=0}^{\max(n,m)} (\mathcal{Q}_i + \mathcal{P}_i) t^i \quad \text{and} \quad \mathcal{Q}(t)\mathcal{P}(t) = \sum_{i=0}^{n+m} \sum_{j+k=i} (\mathcal{Q}_j \mathcal{P}_k) t^i.$$

For simplicity, we also write the product $\mathcal{Q}(t)\mathcal{P}(t)$ as $(\mathcal{Q}\mathcal{P})(t)$. Note also that the variable t is assumed to commute with all elements of $\mathbb{H}$ [62]. The triple $(\mathbb{H}[t], +, \times)$ is a non–commutative ring [48]. It is easily verified that the invertible elements of $\mathbb{H}[t]$ are the constant polynomials.

Definition 1.4 The *conjugate* of $\mathcal{Q}(t) = \mathcal{Q}_n t^n + \cdots + \mathcal{Q}_1 t + \mathcal{Q}_0$ is defined by

$$\mathcal{Q}^*(t) = \mathcal{Q}_n^* t^n + \cdots + \mathcal{Q}_1^* t + \mathcal{Q}_0^*.$$

Below are some of the properties of the conjugate polynomial.

Proposition 1.3. *Given $\mathcal{Q}(t)$, $\mathcal{P}(t) \in \mathbb{H}[t]$ the following properties hold:*

1. $(\mathcal{Q}\mathcal{P})^*(t) = \mathcal{P}^*(t)\mathcal{Q}^*(t)$,
2. $\mathcal{Q}^*(t)\mathcal{Q}(t) = \mathcal{Q}(t)\mathcal{Q}^*(t)$.

Proof: See [67]. ∎

Below we give the definition of the *quasi–norm* in the quaternion polynomial ring.

Definition 1.5 [76] For $\mathcal{Q}(t) \in \mathbb{H}[t]$, the product $\mathcal{Q}(t)\mathcal{Q}^*(t)$ is called the *quasi–norm* of $\mathcal{Q}(t)$, denoted by $N(\mathcal{Q}(t))$. Note that $N(\mathcal{Q}(t))$ is a polynomial with real coefficients.

Remark 1.3. Using the equivalent form of $\mathcal{Q}(t) = u(t) + \mathbf{i}\, v(t) + \mathbf{j}\, p(t) + \mathbf{k}\, q(t)$, we have

$$N(\mathcal{Q}(t)) = u^2(t) + v^2(t) + p^2(t) + q^2(t).$$

Note that $N(\mathcal{Q}(t))$ is often denoted with $|\mathcal{Q}(t)|^2$.

Example 1.2.1 If $\mathcal{Q}(t) = t^2 - \mathbf{j}\, t + \mathbf{i}$, then $\mathcal{Q}^*(t) = t^2 + \mathbf{j}\, t - \mathbf{i}$ and the quasi–norm is

$$N(\mathcal{Q}(t)) = \mathcal{Q}(t)\mathcal{Q}^*(t) = t^4 + t^2 + 1.$$

The equivalent form $\mathcal{Q}(t) = t^2 + \mathbf{i} - \mathbf{j}\, t$ likewise gives $N(\mathcal{Q}(t)) = t^4 + 1 + t^2$.

Definition 1.6 For a polynomial

$$\mathcal{Q}(t) = \sum_{i=0}^{n} \mathcal{Q}_i t^i \in \mathbb{H}[t]$$

and an element $\mathcal{R} \in \mathbb{H}$ we define the *evaluation* of $\mathcal{Q}(t)$ at $t = \mathcal{R}$ to be the element

$$\mathcal{Q}(\mathcal{R}) = \sum_{i=0}^{n} \mathcal{Q}_i \mathcal{R}^i \in \mathbb{H}.$$

It is important to observe that if we have $\mathcal{H}(t) = \mathcal{F}(t)\mathcal{G}(t) \in \mathbb{H}[t]$ for $\mathcal{F}(t), \mathcal{G}(t) \in \mathbb{H}[t]$, it does not follow that $\mathcal{H}(\mathcal{R}) = \mathcal{F}(\mathcal{R})\mathcal{G}(\mathcal{R})$ for $\mathcal{R} \in \mathbb{H}$, i.e., evaluation at $t = \mathcal{R}$ is in general not a ring homomorphism from $\mathbb{H}[t]$ to $\mathbb{H}$. The following example illustrates this special feature of non–commutative division rings.

Example 1.2.2 Consider the quaternion polynomials $\mathcal{Q}(t) = t - \mathbf{k}$, $\mathcal{P}(t) = t - \mathbf{j}$. Then, we define

$$\mathcal{H}(t) := \mathcal{Q}(t)\mathcal{P}(t) = (t - \mathbf{k})(t - \mathbf{j}) = t^2 - (\mathbf{j} + \mathbf{k})\, t - \mathbf{i}.$$

The value of $\mathcal{H}(t)$ at $t = \mathbf{k}$ is

$$\mathcal{H}(\mathbf{k}) = \mathbf{k}^2 - (\mathbf{j} + \mathbf{k})\, \mathbf{k} - \mathbf{i} = -1 - \mathbf{i},$$

but $\mathcal{H}(\mathbf{k}) \neq \mathcal{Q}(\mathbf{k})\mathcal{P}(\mathbf{k}) = (\mathbf{k} - \mathbf{k})(\mathbf{k} - \mathbf{j}) = 0$. Note, however, that for $t = \mathbf{j}$ we have $\mathcal{H}(\mathbf{j}) = \mathbf{j}^2 - (\mathbf{j} + \mathbf{k})\mathbf{j} - \mathbf{i} = 0$ and $\mathcal{Q}(\mathbf{j})\mathcal{P}(\mathbf{j}) = (\mathbf{j} - \mathbf{k})(\mathbf{j} - \mathbf{j}) = 0$. In this case, the values of $\mathcal{H}(t)$ and $\mathcal{Q}(t)\mathcal{P}(t)$ agree, since $t = \mathbf{j}$ is - as we will see - a *root* of $\mathcal{H}(t)$.

Proposition 1.4. *Let $\mathcal{Q}(t)$, $\mathcal{P}(t) \in \mathbb{H}[t]$ and $\mathcal{C} \in \mathbb{H}$. Then the following hold:*

1. $(\mathcal{Q} + \mathcal{P})(\mathcal{C}) = \mathcal{Q}(\mathcal{C}) + \mathcal{P}(\mathcal{C})$,
2. *If $\mathcal{Q}(t) = \mathcal{Q}_n t^n$, where $\mathcal{Q}_n \in \mathbb{H}$ and $n \in \mathbb{N}$ then $(\mathcal{Q}\mathcal{P})(\mathcal{C}) = \mathcal{Q}_n \mathcal{P}(\mathcal{C})\mathcal{C}^n$.*

Proof: See [43]. ∎

Definition 1.7 Let $\mathcal{Q}(t)$, $\mathcal{P}(t) \in \mathbb{H}[t]$. The polynomial $\mathcal{Q}(t)$ is called a *left divisor* (or *left factor*) of $\mathcal{P}(t)$ if there exists a polynomial $\mathcal{D}(t) \in \mathbb{H}$ such that

$$\mathcal{P}(t) = \mathcal{Q}(t)\mathcal{D}(t).$$

The definition of *right divisor* (or *right factor*) is analogous. A polynomial $\mathcal{Q}(t)$ is called a *divisor* of a $\mathcal{P}(t)$ if it is both a *right* and a *left* divisor of $\mathcal{P}(t)$.

Definition 1.8 [62] An element $\mathcal{C} \in \mathbb{H}$ is said to be a *zero* or *root* of $\mathcal{Q}(t) \in \mathbb{H}[t]$ if $\mathcal{Q}(\mathcal{C}) = 0$.

The importance of the fact that evaluation does not preserve multiplication is apparent in the consideration of roots. If $\mathcal{R} \in \mathbb{H}$ with $\mathcal{F}(\mathcal{R}) = 0$, we cannot conclude that $\mathcal{R}$ is also a root of the product $\mathcal{F}(t)\mathcal{G}(t)$ for any $\mathcal{G}(t)$. For instance, in Example 1.2.2, the fact that $\mathbf{k}$ is a root of $\mathcal{Q}(t)$ does not imply that $\mathbf{k}$ is also a root of $\mathcal{H}(t) = \mathcal{Q}(t)\mathcal{P}(t)$.

The next proposition establishes a connection between zeros and right divisors.

Proposition 1.5. *An element $\mathcal{C} \in \mathbb{H}$ is a root of a non–zero polynomial $\mathcal{Q}(t) \in \mathbb{H}[t]$ if and only if the polynomial $t - \mathcal{C}$ is a right divisor of $\mathcal{Q}(t)$.*

Proof: See [62]. ∎

Remark 1.4. In view of the above proposition, the roots of a quaternion polynomial are often called *right roots*.

Note that, in the non–commutative ring $\mathbb{H}$ the roots of a left divisor $\mathcal{D}(t)$ are not necessarily roots of $\mathcal{Q}(t)$. The following proposition shows that there is a relation between the zeros of a polynomial and the zeros of its left divisors. Indeed, as we shall see, if $\mathcal{Q}(t) = \mathcal{P}(t)\mathcal{G}(t) \in \mathbb{H}[t]$ and $\mathcal{C} \in \mathbb{H}$ is a zero of $\mathcal{Q}(t)$ but is not a zero of $\mathcal{G}(t)$, then its left divisor $\mathcal{P}(t)$ must have a zero that it is *similar* (or *conjugate*) to $\mathcal{C}$. Recall that $\mathbb{H}[t]$ is a division ring, and any non–zero element $\mathcal{C}$ has a unique inverse, denoted by $\mathcal{C}^{-1}$.

Proposition 1.6. *Let $\mathcal{Q}(t) = \mathcal{P}(t)\mathcal{G}(t) \in \mathbb{H}[t]$ and $\mathcal{C} \in \mathbb{H}$ such that $\mathcal{A} := \mathcal{G}(\mathcal{C}) \neq 0$. Then*

$$\mathcal{Q}(\mathcal{C}) = \mathcal{P}(\mathcal{A}\mathcal{C}\mathcal{A}^{-1})\,\mathcal{G}(\mathcal{C}).$$

In particular, if $\mathcal{C}$ is a root of $\mathcal{Q}(t)$ but not of $\mathcal{G}(t)$, then $\mathcal{A}\mathcal{C}\mathcal{A}^{-1}$ is a root of $\mathcal{P}(t)$.

Proof: See [62]. ∎

By the above proposition we straightway have the following corollary.

Corollary 1.2. *Let* $\mathcal{Q}(t) = \mathcal{P}(t)\mathcal{G}(t) \in \mathbb{H}[t]$ *and* $\mathcal{C} \in \mathbb{H}$. *Then* $\mathcal{C}$ *is a root of* $\mathcal{Q}(t)$ *if and only if either* $\mathcal{C}$ *is a root of* $\mathcal{G}(t)$ *or* $\mathcal{A}\mathcal{C}\mathcal{A}^{-1}$ *is a root of* $\mathcal{P}(t)$.

Corollary 1.3. *If* $\mathcal{C} \in \mathbb{H}$ *is a root of* $\mathcal{G}(t) \in \mathbb{H}$, *then* $\mathcal{C}$ *is a root of* $\mathcal{P}(t)\mathcal{G}(t)$, *for any* $\mathcal{P}(t) \in \mathbb{H}[t]$.

We denote the set of all zeros of a polynomial $\mathcal{Q}(t)$ by $Z(\mathcal{Q})$. By [72], we have the following corollary, which characterizes the set of zeros of a polynomial when it is specified as the product of two other polynomials.

Corollary 1.4. *Let* $\mathcal{Q}(t) = \mathcal{P}(t)\mathcal{G}(t) \in \mathbb{H}[t]$. *Then*

$$Z(\mathcal{Q}) = Z(\mathcal{G}) \cup \{\, \mathcal{C} \in \mathbb{H} : \mathcal{G}(\mathcal{C}) \neq 0 \ \text{ and } \ \mathcal{G}(\mathcal{C})\mathcal{C}\mathcal{G}(\mathcal{C})^{-1} \in Z(\mathcal{P}) \,\}.$$

Corollary 1.5. *If* $\mathcal{P}(\mathcal{C}) \neq 0$ *is a unit in* $\mathbb{H}$ *then*

$$\mathcal{Q}(\mathcal{C})\mathcal{P}(\mathcal{C}) = \mathcal{Q}(\mathcal{P}(\mathcal{C})\mathcal{C}\mathcal{P}(\mathcal{C})^{-1})\mathcal{P}(\mathcal{C}).$$

Corollary 1.5 provides another way to evaluate $\mathcal{Q}(\mathcal{C})$ without the need to explicitly write $\mathcal{Q}(t) = \mathcal{P}(t)\mathcal{G}(t)$ in power form (this corollary is valid for any division ring).

Example 1.2.3 For $\mathcal{Q}(t) = t - \mathbf{k}$ and $\mathcal{P}(t) = t - \mathbf{j}$, consider the quaternion polynomial $\mathcal{H}(t) = \mathcal{Q}(t)\mathcal{P}(t)$ as in Example 1.2.2. We wish to evaluate $\mathcal{H}(\mathbf{k})$ and $\mathcal{H}(\mathbf{j})$. Using Corollary 1.3, we have $\mathcal{H}(\mathbf{j}) = 0$ since $\mathcal{P}(\mathbf{j}) = 0$. Now

$$\mathcal{P}(\mathbf{k}) = \mathbf{k} - \mathbf{j} \quad \text{and} \quad \mathcal{P}(\mathbf{k})^{-1} = (\mathbf{k} - \mathbf{j})^{-1} = \frac{\mathbf{j} - \mathbf{k}}{2},$$

and using Corollary 1.5 we obtain

$$\mathcal{P}(\mathbf{k})\,\mathbf{k}\,\mathcal{P}(\mathbf{k})^{-1} = (\mathbf{k} - \mathbf{j})\,\mathbf{k}\,(\mathbf{k} - \mathbf{j})^{-1} = -\mathbf{j}$$

and

$$\mathcal{H}(\mathbf{k}) = \mathcal{Q}((\mathbf{k} - \mathbf{j})\,\mathbf{k}\,(\mathbf{k} - \mathbf{j})^{-1})\mathcal{P}(\mathbf{k}) = \mathcal{Q}(-\mathbf{j})\mathcal{P}(\mathbf{k}) = -1 - \mathbf{i},$$

in agreement with Example 1.2.2.

To illustrate Proposition 1.6, we present the following example.

Example 1.2.4 Let $\mathcal{Q}(t) = \mathcal{P}(t)\mathcal{G}(t)$ where $\mathcal{P}(t) = t - 2\,\mathbf{i}$ and $\mathcal{G}(t) = t + \mathbf{j}$. An easy calculation shows $\mathcal{Q}(2\,\mathbf{i}) = -4\mathbf{k} \neq 0$ even though $\mathcal{P}(2\,\mathbf{i}) = 0$. Thus, $2\,\mathbf{i}$ is not a root of $\mathcal{Q}(t)$ even though $t - 2\,\mathbf{i}$ is a factor of $\mathcal{Q}(t)$. On the other hand, $\mathcal{Q}(-\mathbf{j}) = 0$ and $-\mathbf{j}$ is a root of $\mathcal{Q}(t)$ since $\mathcal{G}(-\mathbf{j}) = 0$, i.e., $-\mathbf{j}$ is a root of the right factor.

1.3 Roots of quaternion polynomials

Because of the non–commutative nature of quaternion polynomials, finding their roots is a challenging problem [38, 47, 50, 52, 57, 65, 69, 71, 72, 76]. In the early 1940s, Niven and Eilenberg [64, 65, 15] dealt with the problem of finding such zeros, and formulated the Fundamental Theorem of Algebra for quaternions. Surprisingly, in contrast to complex polynomials, quaternion polynomials may have infinitely many zeros. Consider, for instance, the polynomial $\mathcal{Q}(t) = t^2 + 1$ over $\mathbb{H}$. Then $\mathcal{Q}(\mathbf{i}) = \mathcal{Q}(\mathbf{j}) = \mathcal{Q}(\mathbf{k}) = 0$. Hence $\mathcal{Q}(t)$ has at least three roots, even though it is only quadratic. In fact, for any unit quaternion $\mathcal{C} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k} \in \mathbb{H}$ with $c_1^2 + c_2^2 + c_3^2 = 1$, we have $\mathcal{C}^2 = -(c_1^2 + c_2^2 + c_3^2) = -1$, so $\mathcal{C}$ is also a root of $\mathcal{Q}(t)$. Such quaternions are infinite in number, corresponding to points on the unit sphere in $\mathbb{R}^3$. Hence, the polynomial $\mathcal{Q}(t) = t^2 + 1$ has infinitely many roots.

Clearly, the root structure of polynomials with coefficients in a non–commutative division ring is quite different from that of polynomials with coefficients in a commutative ring. Research on polynomials over non–commutative rings, and especially on the quaternion polynomial ring, has attracted growing interest in recent years, resulting in a number of important and unexpected results. Some of these results are summarized below, to establish the context for results derived in subsequent chapters.

The following theorem gives a more precise result for the number of roots of such polynomials.

Theorem 1.1. [42, 62] *Let $\mathcal{Q}(t) \in \mathbb{H}[t]$ be of degree n. Then, all roots of $\mathcal{Q}(t)$ belong to at most n conjugacy classes of $\mathbb{H}$. Furthermore, the number of roots of $\mathcal{Q}(t)$ is at most n, or infinite.*

Theorem 1.2. [42] *If $\mathcal{Q}(t) \in \mathbb{H}[t]$ is of degree n and has two distinct roots in a conjugacy class of $\mathbb{H}$, it has infinitely–many zeros in that class. In particular, all elements belonging to that class are zeros of $\mathcal{Q}(t)$.*

Remark 1.5. Recall that both $\mathcal{Q}$ and $\mathcal{Q}^*$ are roots of the characteristic polynomial (1.2) of a given quaternion $\mathcal{Q}$. Thus, by the above theorem, all elements of the equivalence class $[\,\mathcal{Q}\,]$ are solutions of the characteristic polynomial $\varphi_{\mathcal{Q}}(t)$ of $\mathcal{Q}$. This polynomial is also called the *minimal* polynomial of the equivalence class $[\,\mathcal{Q}\,]$. Clearly, by definition we have

$$\varphi_{\mathcal{Q}}(\mathcal{Z}) = 0 \quad \Longleftrightarrow \quad \mathcal{Z} \in [\,\mathcal{Q}\,].$$

Pogorui and Shapiro [68] proved that any quaternion polynomial, has two types of roots which are given below.

Definition 1.9 Suppose that $\mathcal{C}_0 \in \mathbb{H}$ is a root of the quaternion polynomial $\mathcal{Q}(t)$. If $\mathcal{C}_0 \notin \mathbb{R}$ and has the property that $\mathcal{Q}(\mathcal{C}) = 0$ for all $\mathcal{C} \in [\,\mathcal{C}_0\,]$ we say that $\mathcal{C}_0$ is a *spherical* root of $\mathcal{Q}(t)$. In that case, we say that $\mathcal{C}_0$ is a *generator* of the spherical root $[\,\mathcal{C}_0\,]$. Otherwise (i.e., $\mathcal{C}_0 \in \mathbb{R}$ or it is not a spherical root), it is called an *isolated* root of $\mathcal{Q}(t)$. The number of zeros of $\mathcal{Q}(t)$ will be defined as the number of equivalence classes which possess at least one zero of $\mathcal{Q}(t)$, i.e., the number of equivalence classes of spherical roots, plus the number of isolated roots.

Remark 1.6. Note that, if $\mathcal{C}_0$ is a zero of $\mathcal{Q}(t)$, then either all elements in $[\,\mathcal{C}_0\,]$ are zeros or only $\mathcal{C}_0$ is a zero. For instance, in the case of a quaternion polynomial $\mathcal{Q}(t)$ with real coefficients (i.e. a real polynomial), if there exists $\mathbf{z}_0 \in \mathbb{C} \setminus \mathbb{R}$ such that $\mathcal{Q}(\mathbf{z}_0) = 0$, then $\mathbf{z}_0$ is definitely a spherical root. Thus, all complex roots of real polynomials are spherical roots of the corresponding quaternion polynomials.

Pogorui and Shapiro [68] also showed that the total number of isolated zeros plus twice the number of the spherical zeros cannot exceed the degree of the polynomial. A more precise statement will be presented in the next subsection. The following examples illustrate this result. The set of zeros of each polynomial is not empty and the number of zeros does not exceed its degree.

Example 1.3.1 [49] Consider the polynomial

$$\mathcal{Q}(t) = t^6 + \mathbf{j}\,t^5 + \mathbf{i}\,t^4 - t^2 - \mathbf{j}\,t - \mathbf{i}$$

of degree $n = 6$. Janovská and Opfer [49] show that $\mathcal{Q}(t)$ has the four isolated roots

$$t_1 = 1, \quad t_2 = -1, \quad t_3 = \frac{1}{2}(1 - \mathbf{i} - \mathbf{j} - \mathbf{k}), \quad t_4 = \frac{1}{2}(-1 + \mathbf{i} - \mathbf{j} - \mathbf{k})$$

and the single spherical root $t_5 = [\,\mathbf{i}\,]$.

Example 1.3.2 [68] Consider the polynomials

$$\mathcal{Q}_1(t) = t^5 + \mathbf{i}\, t^4 + (\mathbf{j}+1)\, t^3 + (\mathbf{k}+1)\, t^2 + \mathbf{j}\, t + \mathbf{k} \quad \text{and} \quad \mathcal{Q}_2(t) = (t^2+1)^2.$$

In [68] it is found that $\mathcal{Q}_1(t)$ has the isolated roots

$$t_1 = \frac{1}{\sqrt{2}}(1-\mathbf{j}), \quad t_2 = -\frac{1}{\sqrt{2}}(1-\mathbf{j}), \quad t_3 = \mathbf{k}$$

together with the spherical root $t_4 = [\,\mathbf{i}\,]$. $\mathcal{Q}_2(t)$, on the other hand, has only the spherical root $t = [\,\mathbf{i}\,]$.

Topuridze [76] focuses on the study of monic quaternion polynomials of degree n. Using the characteristic polynomial (1.2) of a quaternion, he proved the following results.

Proposition 1.7. [76] *Let $\mathcal{Q}(t) = t^n + \cdots + \mathcal{Q}_1 t + \mathcal{Q}_0$ be a quaternion polynomial, and $\mathcal{C}$ a given quaternion. Then either $\varphi_{\mathcal{C}}(t)$ divides $\mathcal{Q}(t)$ and the whole equivalence class $[\,\mathcal{C}\,]$ is a spherical root of $\mathcal{Q}(t)$, or there is no more than one root of $\mathcal{Q}(t)$ in $[\,\mathcal{C}\,]$.*

Corollary 1.6. [76] *The set of roots of $\mathcal{Q}(t)$ is infinite if and and only if there exists $\mathcal{A} \in \mathbb{H}$ such that the characteristic polynomial of $\mathcal{A}$ divides $\mathcal{Q}(t)$.*

The following result emphasizes that the *real* polynomials exhibit a very different behavior over the ring $\mathbb{H}$, since they may have infinitely–many roots.

Theorem 1.3. [76] *Let $\mathcal{Q}(t)$ be a monic quaternion polynomial with real coefficients. If $\mathcal{Q}(t)$ has at least one non–real root, then it has infinitely many quaternion roots. In particular, the zero set of a real monic polynomial over $\mathbb{H}$ is finite if and only if all roots of the polynomial are real.*

Several authors have recently proposed algorithms to systematically compute the roots of quaternion polynomials — including Serodio, Pereira, and Vitoria [71], Pumplun and Walcher [69], Pogorui and Shapiro [68], Gentili and Stoppato [38], Janovská and Opfer [49], Feng and Zhao [37], and Kalantari [57]. The methods in these papers deal with polynomials in which the quaternion coefficients are all on the left or the right of the powers of t. Recall that in the present thesis we focus on the former case.

In the special case of monic quadratic quaternion polynomials, Huang and So [47] gave explicit formulas for the roots. A different method was subsequently proposed by Jia, Cheng and Zhao [52]. Recently, in [22] an algorithm was developed based on the scalar–vector representation of quaternions, separating the roots into two classes, *generic* roots (with distinct scalar parts) and *singular* roots (with coincident scalar parts). As we shall see in Chapter 6, the algorithm is used to characterize the root structure of quadratic quaternion polynomials that generate quintic rational rotation–minimizing frame curves.

As we shall see below, the problem of finding the roots of quaternion polynomials is closely related to the factorization problem for $\mathbb{H}[t]$.

1.4 Factorization of quaternion polynomials

In this section we deal with the factorization of quaternion polynomials and summarize some known results. First we shall see that, as in the case of polynomials over fields, any polynomial over division ring can be written as a product of linear polynomials. Recall that the roots of a quaternion polynomial of degree n come from at most n equivalence classes.

Theorem 1.4. *Let $\mathcal{Q}(t) = \mathcal{Q}_n t^n + \cdots + \mathcal{Q}_0$, where $\mathcal{Q}_0, \ldots, \mathcal{Q}_n \in \mathbb{H}$. Then, there exist elements $\mathcal{C}_1, \ldots, \mathcal{C}_n \in \mathbb{H}$ such that*

$$\mathcal{Q}(t) = \mathcal{Q}_n(t - \mathcal{C}_n)(t - \mathcal{C}_{n-1}) \cdots (t - \mathcal{C}_1).$$

Proof: See [38]. ∎

Theorem 1.5. (Gordon-Motzkin) *Let $\mathcal{Q}(t)$ be a monic polynomial of degree n in $\mathbb{H}[t]$. Then, if*

$$\mathcal{Q}(t) = (t - \mathcal{C}_n)(t - \mathcal{C}_{n-1}) \cdots (t - \mathcal{C}_1),$$

where $\mathcal{C}_n, \mathcal{C}_{n-1}, \ldots, \mathcal{C}_1 \in \mathbb{H}$, any root of $\mathcal{Q}(t)$ is conjugate to some $\mathcal{C}_i$.

Proof: See [62, Theorem 16.4]. ∎

Moreover, Serodio and Siu [72] proved that if the polynomial $\mathcal{Q}(t)$ is decomposed into a product of linear factors as above, then the set of roots of $\mathcal{Q}(t)$ is a subset of $[\,\mathcal{C}_1\,] \cup [\,\mathcal{C}_2\,] \cup \cdots \cup [\,\mathcal{C}_n\,]$.

In Theorem 1.4, the element $\mathcal{C}_1$ is a root of $\mathcal{Q}(t)$, but $\mathcal{C}_2, \ldots, \mathcal{C}_{n-1}, \mathcal{C}_n$ are not necessarily roots of $\mathcal{Q}(t)$. In the special case that $\mathcal{Q}(t)$ has finitely–many roots, we can see from [40, Theorem 2.3] the relation between the factors and roots, and moreover we can find all roots of $\mathcal{Q}(t)$ provided we know its factorizations. Gentili and Struppa [40] also show that every quaternion polynomial with coefficients on one side can be written as a product of linear factors and special quadratic real factors, giving another concrete form for the factorization of a quaternion polynomial.

Theorem 1.6. *Let $\mathcal{P}(t) \in \mathbb{H}[t]$ be of degree n. Then there exist numbers $p, m_1, \ldots, m_p \in \mathbb{N}$ and generators $\mathcal{C}_1, \ldots \mathcal{C}_p \in \mathbb{H}$ of the spherical roots $S_1, \ldots, S_p$ of $\mathcal{P}(t)$, such that*

$$\mathcal{P}(t) = (t^2 - 2\,\mathrm{scal}(\mathcal{C}_1)\,t + |\mathcal{C}_1|^2)^{m_1} \cdots (t^2 - 2\,\mathrm{scal}(\mathcal{C}_p)\,t + |\mathcal{C}_p|^2)^{m_p}\,\mathcal{Q}(t),$$

where $\mathcal{Q}(t) \in \mathbb{H}[t]$ has only isolated roots. Moreover, there exists a constant $\mathcal{A} \in \mathbb{H}$ and r distinct 2–dimensional spheres $\tilde{S}_1, \ldots, \tilde{S}_r$ and associated numbers $n_1, \ldots, n_r \in \mathbb{N}$, where $n_1 + \cdots + n_r = n - 2(m_1 + \cdots + m_p)$, together with quaternions $\mathcal{A}_{ij} \in \tilde{S}_i$ for $i = 1, \ldots, r$ and $j = 1, \ldots, n_i$, such that

$$\mathcal{Q}(t) = \mathcal{A}(t-\mathcal{A}_{rn_r})\cdots(t-\mathcal{A}_{r2})(t-\mathcal{A}_{r1})\,\cdots\,(t-\mathcal{A}_{1n_1})\cdots(t-\mathcal{A}_{12})(t-\mathcal{A}_{11})$$

or

$$\mathcal{Q}(t) = \mathcal{A}\left(\prod_{i=1}^{r}\prod_{j=1}^{n_i}(t - \mathcal{A}_{ij})\right). \tag{1.3}$$

Proof: See [40]. ∎

Example 1.4.1 Let $\mathcal{P}(t)$ be a polynomial of degree $m = 8$, with generators of spherical roots $\mathcal{C}_1 = \mathbf{i}$ and $\mathcal{C}_2 = \mathbf{j} + 1$ such that $m_1 = 2$ and $m_2 = 1$ and isolated roots $\mathcal{Q}_1 = \mathbf{j}$ and $\mathcal{Q}_2 = (6\mathbf{i} + 8\mathbf{j})/5$ with $n_1 = n_2 = 1$. According to the theorem, there exist $r = 2$ distinct spheres S_1, S_2 together with $n_1 = 1$ quaternions $\mathcal{A}_{11} \in S_1$ and $n_2 = 1$ quaternions $\mathcal{A}_{21} \in S_2$, such that

$$\mathcal{P}(t) = (t^2 - 2\,\mathrm{scal}(\mathbf{i})\,t + |\mathbf{i}|^2)^2\,(t^2 - 2\,\mathrm{scal}(1+\mathbf{j})\,t + |1+\mathbf{j}|^2)\,(t - \mathcal{A}_{21})\,(t - \mathcal{A}_{11}).$$

- If we set $\mathcal{A}_{11} = \mathcal{Q}_1 = \mathbf{j}$ then $\mathcal{A}_{21} = (\mathcal{Q}_2 - \mathcal{A}_{11})\mathcal{Q}_2(\mathcal{Q}_2 - \mathcal{A}_{11})^{-1}$ and we have

$$\mathcal{A}_{21} = \left(\frac{6\mathbf{i}+8\mathbf{j}}{5} - \mathbf{j}\right)\frac{6\mathbf{i}+8\mathbf{j}}{5}\left(\frac{6\mathbf{i}+8\mathbf{j}}{5} - \mathbf{j}\right)^{-1} = 2\,\mathbf{i}.$$

Hence,

$$\mathcal{P}(t) = (t^2+1)^2(t^2 - 2\,t + 2)(t - 2\mathbf{j})(t - \mathbf{i}).$$

- If $\mathcal{A}_{11} = \mathcal{Q}_2 = (6\mathbf{i} + 8\mathbf{j})/5$ then

$$\begin{aligned} \mathcal{A}_{21} &= (\mathcal{Q}_1 - \mathcal{A}_{11})\mathcal{Q}_1(\mathcal{Q}_1 - \mathcal{A}_{11})^{-1} \\ &= \left(\mathbf{j} - \frac{6\mathbf{i}+8\mathbf{j}}{5}\right)\mathbf{j}\left(\mathbf{j} - \frac{6\mathbf{i}+8\mathbf{j}}{5}\right)^{-1} \\ &= \frac{4\mathbf{i}-3\mathbf{j}}{5} \end{aligned}$$

and

$$\mathcal{P}(t) = (t^2+1)^2(t^2 - 2\,t + 2)\left(t - \frac{4\mathbf{i}-3\mathbf{j}}{5}\right)\left(t - \frac{6\mathbf{i}+8\mathbf{j}}{5}\right).$$

Thus, we have two factorizations of $\mathcal{P}(t)$.

Remark 1.7. As we noticed, the number of factorizations of a quaternion polynomial evidently depends on the number of its isolated roots.

We have said that, unlike the case of complex polynomials, the factorization (1.3) is not unique. However, if a quaternion polynomial $\mathcal{Q}(t)$ has only one zero, its factorization is unique. More precisely, we have the following result.

Theorem 1.7. *The quaternion polynomial*

$$\mathcal{P}(t) = (t - \mathcal{C}_n)(t - \mathcal{C}_{n-1})\cdots(t - \mathcal{C}_1),$$

where $\mathcal{C}_i \in [\,\mathcal{C}_1\,]$ for $i = 1, \ldots, n$ and $\mathcal{C}_{i+1} \neq \mathcal{C}_i^*$ for $i = 1, \ldots, n-1$ has a unique root, equal to $\mathcal{C}_1$. Moreover, the above factorization is the only factorization of the polynomial $\mathcal{P}(t)$.

Proof: See [40]. ∎

In the commutative case, the factors of a polynomial are directly related to its roots and their multiplicities. In the ring $\mathbb{H}[t]$, however, the factorization is not unique and there is no "obvious" way to define the multiplicity of a root. Gentili and Struppa [40] introduced the following definition of multiplicity, for both the spherical and isolated roots of quaternion polynomials.

Definition 1.10 Consider the factorization of the polynomial $\mathcal{P}(t)$ given in Theorem 1.6. Let $\mathcal{C}_i$, $i = 1, \ldots, n$ be the spherical roots corresponding to the factors $q_{\mathcal{C}_i}(t) = t^2 - 2\,\mathrm{scal}(\mathcal{C}_i)\,t + |\mathcal{C}_i|^2$ and $\gamma_1, \ldots, \gamma_r$ be the isolated roots which are similar to $\mathcal{A}_{11}, \ldots,\ \mathcal{A}_{r1}$ respectively. The *multiplicity* of the spherical root $\mathcal{C}_i$, $i = 1, \ldots, n$ is defined to be the integer $2m_i$ and the multiplicity of the isolated root γ_i the number n_i, $i = 1, \ldots, r$.

Thus, if $\mathcal{Q}(t)$ is a quaternion polynomial of degree n, the sum of the multiplicities of its spherical and isolated roots is equal to n. Let $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$ be the isolated roots of $\mathcal{Q}(t)$. Note that $\mathcal{Q}_1$ may belong to the equivalence class of one of the spherical roots, and that each of the quaternions $\mathcal{Q}_2, \ldots, \mathcal{Q}_n$ may also belong to the equivalence class of one of the spherical roots of $\mathcal{Q}(t)$.

Example 1.4.2 [68] Consider the polynomial

$$\mathcal{Q}(t) = t^5 + \mathbf{i}\,t^4 + (\mathbf{j}+1)t^3 + (\mathbf{k}+1)t^2 + \mathbf{j}\,t + \mathbf{k}$$

In [68] it is found that $\mathcal{Q}_1(t)$ has three isolated roots

$$\mathcal{Q}_1(t) = \frac{1}{\sqrt{2}}(1-\mathbf{j}), \quad \mathcal{Q}_2(t) = -\frac{1}{\sqrt{2}}(1-\mathbf{j}), \quad \mathcal{Q}_3(t) = \mathbf{k}$$

and one spherical root $\mathcal{Q}_4(t) = [\,\mathbf{i}\,]$. Although $\mathbf{k} \in [\,\mathbf{i}\,]$, it is considered to be an isolated root. Each isolated root has multiplicity 1, and the spherical root has multiplicity 2.

Example 1.4.3 Let

$$\mathcal{P}(t) = (t-\mathbf{k})(t-\mathbf{j})(t-\mathbf{i}).$$

From the factorization we see directly that $\mathbf{i}$ is a root of $\mathcal{P}(t)$. Since $\mathbf{j}, \mathbf{k}, \in [\,\mathbf{i}\,]$ we note form Theorem 1.7 that $\mathbf{i}$ is the unique root of $\mathcal{P}(t)$. By Definition 1.10, it is of multiplicity 3.

Example 1.4.4 Consider the polynomial $\mathcal{Q}(t) = t^2 + 1$. By Theorem 3 it has infinitely–many roots. One factorization of $\mathcal{Q}(t)$ is

$$\mathcal{Q}(t) = t^2 + 1 = (t + \mathbf{i})(t - \mathbf{i}),$$

and this implies that all roots of $\mathcal{Q}(t)$ are in $[\,\mathbf{i}\,]$ and thus $\mathbf{i}$ has multiplicity 2 as a root of $\mathcal{Q}(t)$.

The next result shows that, for a quaternion polynomial with known spherical and isolated roots of known multiplicities, one can find all of its factorizations. More precisely, one can find all factorizations of all the polynomials that have the prescribed roots and corresponding multiplicities.

Theorem 1.8. *The set of all quaternion polynomials with assigned spherical roots $\mathcal{C}_1, \ldots \mathcal{C}_s$ of multiplicity $2m_1, \ldots 2m_s$ and isolated roots $\mathcal{Q}_1, \ldots \mathcal{Q}_r$ of multiplicity $n_1, \ldots, n_r$ comprises all polynomials $\mathcal{P}(t)$ that can be written in the form*

$$\mathcal{P}(t) = (t^2 - 2\,\mathrm{scal}(\mathcal{C}_1)\,t + |\mathcal{C}_1|^2)^{m_1} \cdots (t^2 - 2\,\mathrm{scal}(\mathcal{C}_s)\,t + |\mathcal{C}_p|^2)^{m_s}\, \mathcal{Q}(t),$$

with

$$\mathcal{Q}(t) = \mathcal{A}(t - \mathcal{A}_{rn_r}) \cdots (t - \mathcal{A}_{r2})(t - \mathcal{A}_{r1}) \cdots (t - \mathcal{A}_{1n_1}) \cdots (t - \mathcal{A}_{12})(t - \mathcal{A}_{11}).$$

Here $\mathcal{A}$ is an arbitrary non–zero constant, $\mathcal{A}_{11} = \mathcal{Q}_1$, and the quaternions $\mathcal{A}_{1j}$ for $j = 2, \ldots, n_1$ are freely chosen in $[\,\mathcal{Q}_1\,]$ such that $\mathcal{A}_{1,j+1} \neq \mathcal{A}^*_{1j}$ for $j = 1, \ldots n - 1$, and in general for $i = 2, \ldots r$ we have

$$\mathcal{A}_{i1} = [(\mathcal{F}_{i-1} \cdots\ \mathcal{F}_1)(\mathcal{Q}_i)]\, \mathcal{Q}_i\, [(\mathcal{F}_{i-1} \cdots \mathcal{F}_1)(\mathcal{Q}_i)]^{-1}$$

where

$$\mathcal{F}_k(t) = (t - \mathcal{A}_{kn_k}) \cdots\ (t - \mathcal{A}_{k2})(t - \mathcal{A}_{k1})$$

for $k = 1, \ldots r$ and $\mathcal{A}_{ij}$ for $j = 2, \ldots, n_i$ are freely chosen in the equivalence class $[\,\mathcal{Q}_i\,]$ such that $\mathcal{A}_{i,j+1} \neq \mathcal{A}^*_{ij}$ for $j = 1, \ldots, n_i - 1$.

Proof: See [40]. ∎

Note that in [38, 39, 40, 41] the results are presented for quaternion polynomials with the powers of the variable on the left, and hence the roots considered are left roots. The above results have been adjusted to the case where the powers of the variable are to the right of the quaternion coefficients, and the roots of the quaternion polynomials are therefore right roots.

If a factorization of a quaternion polynomial has been constructed, the following result shows how its isolated roots can be determined from this factorization.

Theorem 1.9. [40] *Let $\mathcal{P}(t)$ be a polynomial without spherical roots and let*

$$\mathcal{P}(t) = (t - \mathcal{A}_n)(t - \mathcal{A}_{n-1}) \cdots (t - \mathcal{A}_1)$$

be one of its factorizations. Then, the roots of $\mathcal{P}(t)$ can be obtained from $\mathcal{A}_1, \ldots, \mathcal{A}_n$ as follows. Clearly, $\mathcal{A}_1$ is a root. From $\mathcal{A}_1$ and $\mathcal{A}_2$ yields the root

$$\mathcal{A}_2^{(1)} = (\mathcal{A}_2^* - \mathcal{A}_1)^{-1} \mathcal{A}_2 \, (\mathcal{A}_2^* - \mathcal{A}_1).$$

In general, setting

$$\mathcal{A}_s^{(j)} = (\mathcal{A}_s^{(j-1)*} - \mathcal{A}_{s-j})^{-1} \mathcal{A}_s^{(j-1)} \, (\mathcal{A}_s^{(j-1)*} - \mathcal{A}_{s-j})$$

for $s = 1, \ldots, n$ and $j = 1, \ldots, s-1$, we obtain that the roots of $\mathcal{P}(t)$ are given by

$$\mathcal{A}_s^{(s-1)} = (\mathcal{A}_s^{(s-2)*} - \mathcal{A}_1)^{-1} \mathcal{A}_s^{(s-2)} \, (\mathcal{A}_s^{(s-2)*} - \mathcal{A}_1).$$

Example 1.4.5 Let $\mathcal{P}(t) = (t - \mathbf{i})(t - 2\,\mathbf{k})(t - \mathbf{j})$. By the above theorem, we have:

- $\mathcal{A}_1 = \mathbf{j}$ is a root since $t - \mathbf{j}$ is a right factor;
- $\mathcal{A}_2^{(1)} = (\mathcal{A}_2^* - \mathcal{A}_1)^{-1} \mathcal{A}_2 \, (\mathcal{A}_2^* - \mathcal{A}_1) = (-2\mathbf{k} - \mathbf{j})^{-1} 2\,\mathbf{k}(-2\,\mathbf{k} - \mathbf{j}) = (8\,\mathbf{j} + 6\,\mathbf{k})/5$,
- $\mathcal{A}_3^{(2)} = (\mathcal{A}_3^{(1)*} - \mathcal{A}_1)^{-1} \mathcal{A}_3^{(1)} (\mathcal{A}_3^{(1)*} - \mathcal{A}_1)$, where

  $$\mathcal{A}_3^{(1)} = (\mathcal{A}_3^* - \mathcal{A}_2)^{-1} \mathcal{A}_3 (\mathcal{A}_3^* - \mathcal{A}_2) = (-3\mathbf{i} + 4\mathbf{k})/5,$$

and hence

$$\mathcal{A}_3^{(2)} = \left(\frac{3\,\mathbf{i} - 4\,\mathbf{k} - 5\,\mathbf{j}}{5}\right)^{-1} \frac{-3\,\mathbf{i} + 4\,\mathbf{k}}{5} \left(\frac{3\,\mathbf{i} - 4\,\mathbf{k} - 5\,\mathbf{j}}{5}\right) = \mathbf{j}.$$

Remark 1.8. We see that $\mathcal{A}_1 = \mathcal{A}_3^{(2)} = \mathbf{j}$, i.e., $\mathbf{j}$ is a double root -as expected- since $\mathbf{j}$ is similar to $\mathbf{i}$.

Chapter 2

Complex Roots of Quaternion Polynomials

As we have seen, polynomials with quaternion coefficients have two kinds of roots: isolated and spherical. A spherical root generates a class of roots which contains only two complex numbers $\mathbf{z}$ and its conjugate $\bar{\mathbf{z}}$, and this class can be determined by $\mathbf{z}$. In this chapter we deal with the complex roots of quaternion polynomials. Section 2.1 introduces the Bézout matrices which we shall use for the presentation of our results. In Sections 2.2 and 2.3 we give necessary and sufficient conditions for a quaternion polynomial to have a complex root, a spherical root and a complex isolated root. Finally, in Section 2.4 we compute a bound for the size of the roots of a quaternion polynomial.

2.1 Bézout Matrices

In 1971, Barnett computed the degree (resp. coefficients) of the greatest common divisor of several univariate polynomials with coefficients in an integral domain by means of the rank (resp. linear dependencies of the columns) of several matrices involving their coefficients [2, 3]. In this section we recall a formulation of Barnett's results using Bézout matrices [12] which we shall use for the presentation of our results. We could equally use another formulation of Barnett's results given in [12] or use another approach [36, 56, 78], but we have chosen the formulation with Bézout matrices since it is more simple and quite efficient in computations.

Let F be a field of characteristic zero and $P(x)$, $Q(x)$ polynomials in

$F[x]$ with $d = \max\{\deg P, \deg Q\}$. Consider the polynomial

$$\frac{P(x)Q(y) - P(y)Q(x)}{x - y} = \sum_{i,j=0}^{d-1} c_{i,j} x^i y^j.$$

The *Bézout matrix* associated to $P(x)$ and $Q(x)$ is:

$$\mathrm{Bez}(P, Q) = \begin{pmatrix} c_{0,0} & \cdots & c_{0,d-1} \\ \vdots & \vdots & \vdots \\ c_{d-1,0} & \cdots & c_{d-1,d-1} \end{pmatrix}.$$

The *Bezoutian* associated to $P(x)$ and $Q(x)$ is defined as the determinant of the matrix $\mathrm{Bez}(P, Q)$ and it will be denoted by $\mathrm{bez}(P, Q)$. Let $n = \deg P$, $m = \deg Q$ and p_0 the leading coefficient of $P(x)$. If $n \geq m$, then

$$\mathrm{bez}(P, Q) = (-1)^{n(n-1)/2} p_0^{n-m} R(P, Q),$$

where $R(P, Q)$ is the well known Sylvester resultant of $P(x)$ and $Q(x)$ [3, 46]. Furthermore, we have $\mathrm{bez}(P, Q) = 0$ if and only if $\deg(\gcd(P, Q)) \geq 1$.

Now, let $P(x)$, $Q_1(x), \ldots, Q_k(x)$ be a family of polynomials in $F[x]$ with $n = \deg P$ and $\deg Q_j \leq n - 1$ for every $j \in \{1, \ldots, k\}$. Set

$$\mathbf{B}_P(Q_1, \ldots, Q_k) = \begin{pmatrix} \mathrm{Bez}(P, Q_1) \\ \vdots \\ \mathrm{Bez}(P, Q_k) \end{pmatrix}.$$

We have the following formulation of Barnett's theorem.

Lemma 2.1. *Let* $D = \gcd(P, Q_1, \ldots, Q_k))$. *Then*

$$\deg D = n - \operatorname{rank} \mathbf{B}_P(Q_1, \ldots, Q_k).$$

Proof: See [12, Theorem 3.2]. ∎

Moreover, the matrix $\mathbf{B}_P(Q_1, \ldots, Q_t)$ can provide the greatest common divisor of $P(x)$, $Q_1(x), \ldots, Q_k(x)$ [12, Theorem 3.4].

2.2 Complex Roots

In this section we give a necessary and sufficient condition for a quaternion polynomial to have a complex root in terms of their coefficients.

Let $\mathcal{Q}(t) \in \mathbb{H}[t] \setminus \mathbb{C}[t]$ be a monic polynomial with $\deg \mathcal{Q}(t) = n \geq 1$. Then, there exist $\mathbf{f}(t)$, $\mathbf{g}(t) \in \mathbb{C}[t]$ with $\mathbf{f}(t)\,\mathbf{g}(t) \neq 0$ such that

$$\mathcal{Q}(t) = \mathbf{f}(t) + \mathbf{k}\,\mathbf{g}(t).$$

We write

$$\mathbf{f}(t) = f_1(t) + \mathbf{i}\,f_2(t) \quad \text{and} \quad \mathbf{g}(t) = g_1(t) + \mathbf{i}\,g_2(t),$$

where $f_1(t), f_2(t), g_1(t), g_2(t) \in \mathbb{R}[t]$. Since $\mathcal{Q}(t)$ is monic of degree n, we have $\deg f_1 = \deg \mathbf{f} = n$ and $\deg f_2$, $\deg g_1$, $\deg g_2$ are smaller than n.

Set

$$D(t) = \gcd(f_1(t), f_2(t), g_1(t), g_2(t)) \quad \text{and} \quad \mathbf{E}(t) = \gcd(\mathbf{f}(\mathrm{t}), \mathbf{g}(\mathrm{t})).$$

The polynomial $D(t)$ divides $f_1(t)$ and $f_2(t)$, whence we get that $D(t)$ divides $\mathbf{f}(t)$. Similarly, we deduce that $D(t)$ divides $\mathbf{g}(t)$. It follows that $D(t)$ divides $\mathbf{E}(t)$.

Recall that $\mathcal{Q}_0$ is a root of $\mathcal{Q}(t)$ if and only if $t - \mathcal{Q}_0$ is a right factor of $\mathcal{Q}(t)$, i.e. there exists $\mathcal{G}(t) \in \mathbb{H}[t]$ such that $\mathcal{Q}(t) = \mathcal{G}(t)(t - \mathcal{Q}_0)$. We shall determine the monic right factors of $\mathcal{Q}(t)$ in $\mathbb{C}[t]$ having the highest degree.

Theorem 2.1. *The only monic right factor of $\mathcal{Q}(t)$ in $\mathbb{C}[t]$ having the highest degree is $\mathbf{E}(t)$ and its degree is $n - \mathrm{rankBez}(\mathbf{f}, \mathbf{g})$. Furthermore, if $z \in \mathbb{R}$, then z is a root of $\mathcal{Q}(t)$ if and only if it is a root of $D(t)$.*

Proof: Let $\mathbf{G}(t)$ be a right factor of $\mathcal{Q}(t)$ in $\mathbb{C}[t]$ having the highest degree. Then, there are $\mathbf{a}(t), \mathbf{b}(t) \in \mathbb{C}[t]$ such that $\mathcal{Q}(t) = (\mathbf{a}(t) + \mathbf{k}\,\mathbf{b}(t))\mathbf{G}(t)$, and thus $\mathbf{f}(t) = \mathbf{a}(t)\mathbf{G}(t)$ and $\mathbf{g}(t) = \mathbf{b}(t)\mathbf{G}(t)$. It follows that $\mathbf{G}(t)$ divides $\mathbf{E}(t)$. On the other hand, there are $\mathbf{f}_1(t), \mathbf{g}_1(t) \in \mathbb{C}[t]$ such that $\mathbf{f}(t) = \mathbf{f}_1(t)\mathbf{E}(t)$ and $\mathbf{g}(t) = \mathbf{g}_1(t)\mathbf{E}(t)$. Then $\mathcal{Q}(t) = (\mathbf{f}_1(t) + \mathbf{k}\,\mathbf{g}_1(t))\mathbf{E}(t)$. Since $\mathbf{G}(t)$ divides $\mathbf{E}(t)$, $\mathbf{G}(t)$ and $\mathbf{E}(t)$ are monic and $\mathbf{G}(t)$ is a highest degree right factor of $\mathcal{Q}(t)$ in $\mathbb{C}[t]$, we deduce that $\mathbf{G}(t)$=$\mathbf{E}(t)$. By Lemma 2.1, we have $\deg \mathbf{E}(t) = n - \mathrm{rankBez}(\mathbf{f}, \mathbf{g})$.

Suppose that $\mathbf{z} \in \mathbb{R}$. Then $\mathcal{Q}(\mathbf{z}) = 0$ if and only if $\mathbf{f}(\mathbf{z}) = \mathbf{g}(\mathbf{z}) = 0$. Since $f_1(\mathbf{z}), f_2(\mathbf{z}), g_1(\mathbf{z}), g_2(\mathbf{z}) \in \mathbb{R}$, we obtain that $\mathbf{f}(\mathbf{z}) = \mathbf{g}(\mathbf{z}) = 0$ is equivalent to $f_1(\mathbf{z}) = f_2(\mathbf{z}) = g_1(\mathbf{z}) = g_2(\mathbf{z}) = 0$, and so $D(\mathbf{z}) = 0$. ∎

Corollary 2.1. *The polynomial $\mathcal{Q}(t)$ has a complex root if and only if we have $R(\mathbf{f}, \mathbf{g}) = 0$ or equivalently* $\mathrm{bez}(\mathbf{f}, \mathbf{g}) = 0$.

Proof: By Theorem 2.1, $\mathcal{Q}(t)$ has a complex root if and only $\deg \mathbf{E}(t) > 0$. Further, we have that $\deg \mathbf{E}(t) > 0$ if and only if $R(\mathbf{f}, \mathbf{g}) = 0$ which is equivalent to $\mathrm{bez}(\mathbf{f}, \mathbf{g}) = 0$. ∎

Corollary 2.2. *The polynomial $\mathcal{Q}(t)$ has at most $n - \mathrm{rankBez}(\mathbf{f}, \mathbf{g})$ complex roots.*

Example 2.2.1 Consider the polynomial

$$\mathcal{Q}(t) = t^3 + (-\mathbf{i} - 2\mathbf{j} - \mathbf{k})\, t^2 + (2\mathbf{i} + \mathbf{j} - 2\mathbf{k})\, t + 2.$$

Write $\mathcal{Q}(t) = \mathbf{f}(t) + \mathbf{k}\, \mathbf{g}(t)$, where

$$\mathbf{f}(t) = t^3 - \mathbf{i}\, t^2 + 2\mathbf{i}\, t + 2 \quad \text{and} \quad \mathbf{g}(t) = -(2\mathbf{i} + 1)\, t^2 + (\mathbf{i} - 2)\, t.$$

We have

$$\frac{\mathbf{f}(x)\mathbf{g}(y) - \mathbf{g}(x)\mathbf{f}(y)}{x - y} = (1+2\mathbf{i})(-x^2\, y + \mathbf{i}\, x\, y^2 + \mathbf{i}\, x^2\, y + x\, y + 2\mathbf{i}\, x\, y + 2\, x + 2\, y - 2\, \mathbf{i}).$$

and so we deduce

$$\mathrm{Bez}(\mathbf{f}, \mathbf{g}) = (1 + 2\mathbf{i}) \begin{pmatrix} -2\mathbf{i} & 2 & 0 \\ 2 & 1 + 2\mathbf{i} & \mathbf{i} \\ 0 & \mathbf{i} & -1 \end{pmatrix}.$$

We see that $\mathrm{rankBez}(\mathbf{f}, \mathbf{g}) = 2$, and so Theorem 2.1 implies that $\mathcal{Q}(t)$ has exactly one complex root. In fact, this root is $t = \mathbf{i}$. Furthermore, one of the factorizations of $\mathcal{Q}(t)$ is

$$\mathcal{Q}(t) = (t - 2\,\mathbf{j})(t - \mathbf{k})(t - \mathbf{i}).$$

Since $\mathbf{i}$ is the only complex root of $\mathcal{Q}(t)$, the root $\mathbf{i}$ is isolated.

The case of a quadratic quaternion equation has been studied in [22, 47, 52, 64]. The next corollary provides their solutions in the special case where one of them is complex.

Corollary 2.3. *Let $\mathcal{Q}(t) = t^2 + \mathcal{B}t + \mathcal{C}$ be a quadratic polynomial of $\mathbb{H}[t] \setminus \mathbb{C}[t]$ with no real factor. Set $\mathcal{B} = \mathbf{b}_1 + \mathbf{k}\,\mathbf{c}_1$ and $\mathcal{C} = \mathbf{b}_0 + \mathbf{k}\,\mathbf{c}_0$, where $\mathbf{b}_0, \mathbf{b}_1, \mathbf{c}_0, \mathbf{c}_1 \in \mathbb{C}$. Then $\mathcal{Q}(t)$ has one complex root if and only if*

$$\mathbf{c}_0^2 - \mathbf{c}_0\mathbf{b}_1\mathbf{c}_1 + \mathbf{b}_0\mathbf{c}_1^2 = 0.$$

In this case $\mathbf{c}_0\mathbf{c}_1 \neq 0$, and the roots of $\mathcal{Q}(t)$ are

$$\mathbf{q} = -\frac{\mathbf{c}_0}{\mathbf{c}_1}, \qquad \mathcal{S} = (\mathbf{q} - \mathcal{P}^*)^{-1}\mathcal{P}(\mathbf{q} - \mathcal{P}^*),$$

where $\mathcal{P} = -(\mathbf{b}_0\mathbf{c}_1/\mathbf{c}_0 + \mathbf{k}\,\mathbf{c}_1)$.

Proof: Let $\mathbf{f}(t) = t^2 + \mathbf{b}_1 t + \mathbf{b}_0$ and $\mathbf{g}(t) = \mathbf{c}_1 t + \mathbf{c}_0$. We have

$$R(\mathbf{f}, \mathbf{g}) = \mathbf{c}_0^2 - \mathbf{c}_0\mathbf{b}_1\mathbf{c}_1 + \mathbf{b}_0\mathbf{c}_1^2$$

and by Corollary 2.1, $\mathcal{Q}(t)$ has a complex root if and only if the above quantity is zero.

Suppose now that $\mathcal{Q}(t)$ has a complex root $\mathbf{q}$. If $\mathbf{c}_1 = 0$, then the equality $R(\mathbf{f}, \mathbf{g}) = 0$ implies $\mathbf{c}_0 = 0$ and hence $\mathcal{Q}(t) \in \mathbb{C}[t]$ which is a contradiction. Thus $\mathbf{c}_1 \neq 0$. If $\mathbf{c}_0 = 0$, then we deduce $\mathbf{b}_0 = 0$, and so t is a factor of $\mathcal{Q}(t)$ which is a contradiction. Therefore $\mathbf{c}_0\mathbf{c}_1 \neq 0$.

By Theorem 2.1, we have $\mathbf{g}(\mathbf{q}) = 0$ and $\mathbf{f}(\mathbf{q}) = 0$. It follows that $\mathbf{q} = -\mathbf{c}_0/\mathbf{c}_1$ and $\mathbf{f}(t) = (t - \mathbf{b}_0/\mathbf{q})(t - \mathbf{q})$. Thus, we have the factorization

$$\mathcal{Q}(t) = (t - \mathcal{P})(t - \mathbf{q}),$$

where $\mathcal{P} = -(\mathbf{b}_0\mathbf{c}_1/\mathbf{c}_0 + \mathbf{k}\,\mathbf{c}_1)$. If $\mathcal{P} = \bar{\mathbf{q}}$, then we have $\mathbf{b}_0\mathbf{c}_1/\mathbf{c}_0 + \mathbf{k}\,\mathbf{c}_1 = \bar{\mathbf{c}}_0/\bar{\mathbf{c}}_1$. It follows that $\mathbf{c}_1 = 0$ which is a contradiction. Thus, [72, Lemma 1] yields

$$\mathcal{Q}(t) = (t - (\mathcal{P} - \bar{\mathbf{q}})\mathbf{q}(\mathcal{P} - \bar{\mathbf{q}})^{-1})(t - (\mathbf{q} - \mathcal{P}^*)^{-1}\mathcal{P}(\mathbf{q} - \mathcal{P}^*)).$$

Hence, the other root of $\mathcal{Q}(t)$ is $\mathcal{S} = (\mathbf{q} - \mathcal{P}^*)^{-1}\mathcal{P}(\mathbf{q} - \mathcal{P}^*)$. ■

We finish this section with the following result.

Proposition 2.1. *Let $\mathcal{P}(t)$ be a monic quaternion polynomial of degree n with n distinct complex roots. Then $\mathcal{P}(t)$ is a complex polynomial.*

Proof: We use induction on the degree of $\mathcal{P}(t)$. Let $\deg \mathcal{P}(t) = 1$. Then if $\mathcal{C}$ is a root of $\mathcal{P}(t)$, we have $\mathcal{C} \in \mathbb{C}$ so $\mathcal{P}(t) = t - \mathcal{C} \in \mathbb{C}[t]$. Suppose the proposition holds for every monic quaternion polynomial of degree k with only complex roots. Let now $\mathcal{P}(t)$ be of $\deg \mathcal{P}(t) = k + 1$. We will show that $\mathcal{P}(t)$ must be a complex polynomial. Let $\mathcal{C}_1, \ldots, \mathcal{C}_{k+1}$ be the distinct complex roots of $\mathcal{P}(t)$. Then, we have

$$\mathcal{P}(t) = \mathcal{Q}(t)(t - \mathcal{C}_{k+1}) \tag{2.1}$$

Since $\mathcal{C}_1$ is not a root of the factor $(t - \mathcal{C}_{k+1})$, by Proposition 1.6 we see that the complex value

$$(\mathcal{C}_1 - \mathcal{C}_{k+1})\mathcal{C}_1(\mathcal{C}_1 - \mathcal{C}_{k+1})^{-1}$$

is a root of $\mathcal{Q}(t)$. Similar arguments show that the complex values

$$(\mathcal{C}_2 - \mathcal{C}_{k+1})\mathcal{C}_2(\mathcal{C}_2 - \mathcal{C}_{k+1})^{-1}, \ldots, (\mathcal{C}_k - \mathcal{C}_{k+1})\mathcal{C}_k(\mathcal{C}_k - \mathcal{C}_{k+1})^{-1}$$

must also be roots of $\mathcal{Q}(t)$. It can easily be seen that these complex values are distinct numbers. Since $\mathcal{Q}(t)$ is a quaternion polynomial of degree k with k distinct complex roots, the induction hypothesis indicates that it must be a complex polynomial. But since $\mathcal{C}_{k+1}$ is a complex number, expression (2.1) shows that the polynomial $\mathcal{P}(t)$, of degree $k + 1$, must also be a complex polynomial. ∎

In the case where the roots of $\mathcal{P}(t)$ are not distinct, Proposition 3.1 does not hold, as it can be seen from the following example.

Remark 2.1. Note that in case where the quaternion polynomial has a multiple root the above result is not longer true. For example consider the polynomial $\mathcal{G}(t) = (t - \mathbf{k})(t - \mathbf{i})$. This polynomial has the single complex root $\mathbf{i}$, of multiplicity 2, but it is *not* a complex polynomial.

Corollary 2.4. *If a quadratic polynomial in $\mathbb{H}[t] \setminus \mathbb{C}[t]$ has two distinct roots, then at least one of them is not complex.*

2.3 Spherical and Complex Isolated Roots

In this section, we give necessary and sufficient conditions for a quaternion polynomial to have a spherical root or have a complex isolated root. First, we consider the spherical roots.

Theorem 2.2. *Let* $\mathbf{z} \in \mathbb{C}\backslash\mathbb{R}$ *and* $\mathcal{Q}(t) \in \mathbb{H}[t]$. *The following are equivalent:*
(a) $\mathbf{z}$ *is a spherical root of* $\mathcal{Q}(t)$.
(b) $\mathbf{z}$ *and its conjugate* $\bar{\mathbf{z}}$ *are common roots of* $\mathbf{f}(t)$ *and* $\mathbf{g}(t)$.
(c) $\mathbf{z}$ *is a common root of* $f_1(t), f_2(t), g_1(t), g_2(t)$.

Proof: If $\mathbf{z}$ is a spherical root of $\mathcal{Q}(t)$, then its conjugate $\bar{\mathbf{z}}$ is also a root of $\mathcal{Q}(t)$. Thus, Theorem 2.1 implies that $\mathbf{z}$ and $\bar{\mathbf{z}}$ are common roots of $\mathbf{f}(t)$ and $\mathbf{g}(t)$. If this holds, then the polynomial $(t-\mathbf{z})(t-\bar{\mathbf{z}})$ is a factor of $\mathbf{f}(t)$ and $\mathbf{g}(t)$. It follows that $(t-\mathbf{z})(t-\bar{\mathbf{z}})$ is a factor of $f_1(t), f_2(t), g_1(t), g_2(t)$. Hence $\mathbf{z}$ is a common root of $f_1(t), f_2(t), g_1(t), g_2(t)$. Finally, if $\mathbf{z}$ is a common root of $f_1(t), f_2(t), g_1(t), g_2(t)$, then $\bar{\mathbf{z}}$ is also a common root of $f_1(t), f_2(t), g_1(t), g_2(t)$. Hence $\mathbf{z}$ and $\bar{\mathbf{z}}$ are roots of $\mathcal{Q}(t)$ and so, they define the same spherical root. ■

Corollary 2.5. *If* $\mathcal{Q}(t)$ *has no real factor, then it has only isolated roots.*

Proof: Suppose that $\mathcal{Q}(t)$ has a spherical root $\mathcal{C}$. Then there is a complex number $\mathbf{z} \in [\mathcal{C}]$. It follows that $\mathbf{z}$ is a spherical root of $\mathcal{Q}(t)$ and so, Theorem 2.2(b) implies that $\mathbf{z}$ and its conjugate $\bar{\mathbf{z}}$ are common roots of $\mathbf{f}(t)$ and $\mathbf{g}(t)$. Thus, the real polynomial $(t-\mathbf{z})(t-\bar{\mathbf{z}})$ is a common factor of $\mathbf{f}(t)$ and $\mathbf{g}(t)$. Therefore, $\mathcal{Q}(t)$ has the real factor $(t-\mathbf{z})(t-\bar{\mathbf{z}})$ which is a contradiction. Hence $\mathcal{Q}(t)$ has only isolated roots.

■

Remark 2.2. Since a spherical root of $\mathcal{Q}(t)$ has in its class a number $\mathbf{z} \in \mathbb{C} \setminus \mathbb{R}$, Theorem 2.2 yields that we can find the spherical roots of $\mathcal{Q}(t)$ by computing all the common complex roots of $f_1(t), f_2(t), g_1(t), g_2(t)$.

Theorem 2.3. *Suppose that the quaternion polynomial* $\mathcal{Q}(t)$ *has no real root. The following are equivalent:*
(a) The polynomial $\mathcal{Q}(t)$ *has a spherical root.*
(b) $\deg D(t) > 0$.
(c) $n > \operatorname{rank} \mathbf{B}_{f_1}(f_2, g_1, g_2)$.

Proof: Suppose that $\mathcal{Q}(t)$ has a spherical root $\mathcal{C}$. Let $\mathbf{z}$ and $\bar{\mathbf{z}}$ be the only complex numbers of the class of $\mathcal{C}$. Then we have $\mathcal{Q}(\mathbf{z}) = \mathcal{Q}(\bar{\mathbf{z}})$, whence we get

$$\mathbf{f}(\mathbf{z}) = \mathbf{f}(\bar{\mathbf{z}}) = 0 \quad \text{and} \quad \mathbf{g}(\mathbf{z}) = \mathbf{g}(\bar{\mathbf{z}}) = 0.$$

It follows that the real polynomial $(t - \mathbf{z})(t - \bar{\mathbf{z}})$ divides $\mathbf{f}(\mathbf{z})$ and $\mathbf{g}(\mathbf{z})$ and hence $D(t)$. Therefore $\deg D(t) > 0$.

Conversely, suppose that $\deg D(t) > 0$. Then $D(t)$ has a root $\mathbf{z} \in \mathbb{C}$. If $\mathbf{z} \in \mathbb{R}$, then $\mathbf{z}$ is a common root of $f_1(t), f_2(t), g_1(t), g_2(t)$ and hence $\mathbf{z}$ is a root of $\mathcal{Q}(t)$. Since $\mathcal{Q}(t)$ has no real root we have a contradiction. Thus $\mathbf{z} \notin \mathbb{R}$ and so, its conjugate $\bar{\mathbf{z}}$ is also a root of $D(t)$. It follows that $\mathbf{z}$ and $\bar{\mathbf{z}}$ are roots of $\mathcal{Q}(t)$. By Theorem 2.2, the class of $\mathbf{z}$ is a spherical root of $\mathcal{Q}(t)$.

Finally, by Lemma 2.1 we have

$$\deg D = n - \operatorname{rank} \mathbf{B}_{f_1}(f_2, g_1, g_2)$$

and so, the equivalence of (b) and (c) follows. ∎

Remark 2.3. In the above theorem, the hypothesis that $\mathcal{Q}(t)$ has no real root, implies that $D(t)$ does not have a real root and so, if $\deg D > 0$, then we have that $\deg D$ is even.

Theorem 2.4. *Suppose that the quaternion polynomial $\mathcal{Q}(t)$ has no real root. The following are equivalent:*
(a) The polynomial $\mathcal{Q}(t)$ has an isolated complex root.
(b) $\deg \mathbf{E} > \deg D$.
(c) $\operatorname{rank} \operatorname{Bez}(f, g) < \operatorname{rank} \mathbf{B}_{f_1}(f_2, g_1, g_2)$.

Proof: Let $\mathbf{z}$ be an isolated complex root of $\mathcal{Q}(t)$. By Theorem 2.1, $\mathbf{z}$ is a common root of $\mathbf{f}(t)$ and $\mathbf{g}(t)$. Since the root $\mathbf{z}$ is isolated, $\bar{\mathbf{z}}$ is not a common root of these two polynomials. Thus, the real polynomial $(t-\mathbf{z})(t-\bar{\mathbf{z}})$ is not a common factor of $f_1(t), f_2(t), g_1(t), g_2(t)$. Hence $\mathbf{z}$ is not a root of $D(t)$. Since $D(t)$ divides $\mathbf{E}(t)$, we deduce that $\deg \mathbf{E} > \deg D$.

Conversely, suppose that $\deg \mathbf{E} > \deg D$. Then $\mathbf{E}(t)$ has a complex root $\mathbf{z}$ which is not a root of $D(t)$. If $\mathbf{z}$ is a spherical root, then $\bar{\mathbf{z}}$ is also a common root of $\mathbf{f}(t)$ and $\mathbf{g}(t)$. It follows that $(t - \mathbf{z})(t - \bar{\mathbf{z}})$ is a common factor of $f_1(t), f_2(t), g_1(t), g_2(t)$ and hence divides $D(t)$. Therefore $\mathbf{z}$ is a root of $D(t)$ which is a contradiction. Thus, $\mathbf{z}$ is an isolated root of $\mathcal{Q}(t)$.

By Lemma 2.1, we have

$$\deg D = n - \operatorname{rank} \mathbf{B}_{f_1}(f_2, g_1, g_2) \quad \text{and} \quad \deg \mathbf{E} = n - \operatorname{rankBez}(\mathbf{f}, \mathbf{g}).$$

Thus, we have $\deg \mathbf{E} > \deg D$ if and only if

$$\operatorname{rankBez}(\mathbf{f}, \mathbf{g}) < \operatorname{rank} \mathbf{B}_{f_1}(f_2, g_1, g_2).$$

∎

2.4 Bounds for the Size of the Roots

In [66, Section 4] some bounds for the roots of quaternion polynomials are given. In this section we compute new bounds comparable with that given in [66, Theorem 4.2] and which are better provided some addition hypothesis on the coefficients of the polynomial (see Remark 2.4).

Let

$$\mathcal{Q}(t) = \mathcal{A}_0 t^n + \mathcal{A}_1 t^{n-1} + \cdots + \mathcal{A}_n.$$

be a quaternion polynomial of degree n. We define the *height* of $\mathcal{Q}(t)$ to be the quantity

$$H(\mathcal{Q}) = \max\{1, |\mathcal{A}_1/\mathcal{A}_0|, \ldots, |\mathcal{A}_n/\mathcal{A}_0|\}.$$

We write $\mathcal{Q}(t) = \mathbf{f}(t) + \mathbf{k}\,\mathbf{g}(t)$, where $\mathbf{f}(t), \mathbf{g}(t) \in \mathbb{C}[t]$, and

$$\mathbf{f}(t) = f_1(t) + \mathbf{i}\, f_2(t), \quad \mathbf{g}(t) = g_1(t) + \mathbf{i}\, g_2(t),$$

where $f_1(t), f_2(t), g_1(t), g_2(t) \in \mathbb{R}[t]$. Set

$$H_1 = \min\{H(f), H(g)\} \quad \text{and} \quad H_2 = \min\{H(f_1), H(f_2), H(g_1), H(g_2)\}.$$

Theorem 2.5. *Suppose that the polynomial $\mathcal{Q}(t)$ is monic and $\mathcal{C}$ is a root of $\mathcal{Q}(t)$. If $\mathcal{C}$ is a spherical root, then*

$$|\mathcal{C}| < 1 + H_2^{1/2}.$$

If $\mathcal{C}$ is an isolated complex root, then

$$|\mathcal{C}| < 1 + H_1.$$

In the general case, we have

$$|\mathcal{C}| < 1 + H(\mathcal{Q}).$$

Proof: Suppose that $\mathcal{C}$ is a spherical root of $\mathcal{Q}(t)$. Then there is $\mathbf{z} \in \mathbb{C} \setminus \mathbb{R}$ in the class of $\mathcal{C}$ which is also a root of $\mathcal{Q}(t)$. By Theorem 2.2, $\mathbf{z}$ is a common complex root of $f_1(t), f_2(t), g_1(t), g_2(t)$. Thus, [63, Corollary 3] implies that $|\mathbf{z}| < 1 + H_2^{1/2}$. Since $|\mathcal{C}| = |\mathbf{z}|$, we obtain $|\mathcal{C}| < 1 + H_2^{1/2}$.

Suppose that $\mathcal{C}$ is an isolated root. If $\mathcal{C} \in \mathbb{C}$, then Theorem 2.1 implies that $\mathcal{C}$ is a common root of $\mathbf{f}(t)$ and $\mathbf{g}(t)$. Hence [63, Corollary 2] yields $|\mathcal{C}| < 1 + H_1$.

Suppose next that $\mathcal{C}$ is an isolated non-complex root. If $|\mathcal{C}| \leq 1$, then the result is true. Suppose that $|\mathcal{C}| > 1$. Since $\mathcal{C}$ is a root of $\mathcal{Q}(t)$, there is $\mathcal{G}(t) \in \mathbb{H}[t]$ such that $\mathcal{Q}(t) = \mathcal{G}(t)(t - \mathcal{C})$. Write

$$\mathcal{G}(t) = t^{n-1} + \mathcal{B}_1 t^{n-2} + \cdots + \mathcal{B}_{n-1}.$$

Then

$$\mathcal{Q}(t) = \mathcal{G}(t)(t - \mathcal{C}) = t^n + (\mathcal{B}_1 - \mathcal{C})t^{n-1} + (\mathcal{B}_2 - \mathcal{B}_1\mathcal{C})t^{n-2} + \cdots + \mathcal{B}_{n-1}\mathcal{B}.$$

It follows that

$$\mathcal{A}_1 = \mathcal{B}_1 - \mathcal{C}, \quad \mathcal{A}_2 = \mathcal{B}_2 - \mathcal{B}_1\mathcal{C}, \quad \mathcal{A}_3 = \mathcal{B}_3 - \mathcal{B}_2\mathcal{C}, \ldots, \quad \mathcal{A}_n = \mathcal{B}_{n-1}\mathcal{C}.$$

Let i be the smallest index such that $H(G) = |\mathcal{B}_i|$. Then we have

$$H(Q) \geq |\mathcal{B}_i - \mathcal{B}_{i-1}\mathcal{C}| \geq ||\mathcal{B}_i| - |\mathcal{B}_{i-1}\mathcal{C}|| \geq |H(G) - |\mathcal{B}_{i-1}\mathcal{B}|| > |H(G)(1 - |\mathcal{B}|)|,$$

whence we deduce the result. ∎

Remark 2.4. In case where $\mathcal{A}_0 = 1$, [66, Theorem 4.2] yields that the roots $\mathcal{C}$ of $\mathcal{Q}(t)$ satisfy

$$|\mathcal{C}| \leq \max\{1, \sum_{i=1}^{n} |\mathcal{A}_i|\}.$$

If $\sum_{i=1}^{n} |\mathcal{A}_i| > 1 + H(\mathcal{Q})$, then Theorem 2.5 gives a better bound than the one given in [66, Theorem 4.2].

Corollary 2.6. *Let $\mathcal{Q}(t) \in \mathbb{H}[t] \setminus \mathbb{H}$ be a monic polynomial. Then $\mathcal{Q}(t)$ has at most a finite number of roots $\mathcal{X}$ of the form $\mathcal{X} = x_1 + x_2 i + x_3 j + x_4 k$, where x_1, x_2, x_3, x_4 are integers.*

Chapter 3

Quadratic Quaternion Polynomials

In this chapter we present some results on quadratic quaternion polynomials which are pertinent for the study of rational rotation–minimizing frame curves (RRMF curves). Section 3.1 presents some known results for the factorization of a quadratic quaternion polynomial and Section 3.2 includes conditions in terms of real variables which must be consistent in order for a quadratic equation to have special kinds and specific multitude of roots. Although several methods have been proposed for finding the roots of a quadratic quaternion equation, the scalar–vector algorithm which is demonstrated in Section 3.3 is simpler than others and is used to analyze the root structure of a quadratic quaternion polynomial that generates quintic RRMF curves.

3.1 Factorization of quadratic polynomials

Consider the monic quadratic quaternion polynomial

$$\mathcal{P}(t) = t^2 + \mathcal{B}\,t + \mathcal{C},$$

with at least one of $\mathcal{B}, \mathcal{C}$ not complex. Suppose that the set of roots of $\mathcal{P}(t)$ is infinite. Thus, Corollary 1.6 implies that there is $\mathcal{A} \in \mathbb{H}$ such that its characteristic polynomial $\varphi(t)$ divides $\mathcal{P}(t)$. Since both $\mathcal{P}(t)$ and $\varphi(t)$ are monic quadratic polynomials, this implies that $\mathcal{P}(t) = \varphi(t)$. Hence, every monic quadratic quaternion polynomial with at least one non–real coefficient has at most two roots which are isolated.

Let $\mathcal{C}_1 \in \mathbb{H}$ be a root of $\mathcal{P}(t)$. Then,

$$\mathcal{P}(t) = (t - \mathcal{C}_2)(t - \mathcal{C}_1), \quad \text{where } \ \mathcal{C}_2 \in \mathbb{H}.$$

Suppose first that $\mathcal{P}(t)$ is not a real polynomial.

Then, $\mathcal{C}_1 \neq \mathcal{C}_2^*$ and so, [72] implies that the polynomial $\mathcal{P}(t)$ can also be written in the form

$$\mathcal{P}(t) = \left[t - (\mathcal{C}_2 - \mathcal{C}_1^*)\mathcal{C}_1(\mathcal{C}_2 - \mathcal{C}_1^*)^{-1}\right]\left[t - (\mathcal{C}_1 - \mathcal{C}_2^*)\mathcal{C}_2(\mathcal{C}_1 - \mathcal{C}_2^*)^{-1}\right].$$

From the latter form, we directly obtain that the second root of $\mathcal{P}(t)$ is

$$(\mathcal{C}_1 - \mathcal{C}_2^*)\,\mathcal{C}_2\,(\mathcal{C}_1 - \mathcal{C}_2^*)^{-1}.$$

If $\mathcal{P}(t)$ is a real polynomial, then all its roots lie on the equivalence class $[\,\mathcal{C}_2\,]$, i.e., in the sphere generated by $\mathcal{C}_2$. The following proposition summarizes these results for quadratic quaternion polynomials, indicating the relationship between factors and roots.

Proposition 3.1. *Let $\mathcal{P}(t) = (t - \mathcal{C}_1)(t - \mathcal{C}_2)$ where $\mathcal{C}_1, \mathcal{C}_2 \in \mathbb{H}$. Then, we have:*

1. *If $\mathcal{C}_2$ is not similar to $\mathcal{C}_1$, i.e., $\mathcal{C}_1$ does not belong to the equivalence class of $\mathcal{C}_2$, then $\mathcal{P}(t)$ has two distinct roots, namely*

$$\mathcal{C}_2 \quad \text{and} \quad (\mathcal{C}_2 - \mathcal{C}_1^*)\,\mathcal{C}_1\,(\mathcal{C}_2 - \mathcal{C}_1^*)^{-1}.$$

2. *If $\mathcal{C}_2$ is similar to $\mathcal{C}_1$ but $\mathcal{C}_2 \neq \mathcal{C}_1^*$, then $\mathcal{P}(t)$ has the single root $\mathcal{C}_2$.*

3. *If $\mathcal{C}_2 = \mathcal{C}_1^*$ the zero set of $\mathcal{P}(t)$ is the entire equivalence class of $\mathcal{C}_2$.*

Remark 3.1. From the above, we observe that every quaternion polynomial $\mathcal{P}(t) = t^2 + \mathcal{P}_1 t + \mathcal{P}_0 \in \mathbb{H}[t] \setminus \mathbb{R}[t]$ has either two distinct roots, or one double root.

3.2 Roots of quadratic polynomials

In this section we specifically focus on monic quadratic non–complex polynomials — i.e., polynomials with coefficients in $\mathbb{H} \setminus \mathbb{C}$ — and our goal is to find necessary and sufficient conditions in terms of the coefficients to determine the number and nature of their roots.

Recall that a non–real monic quadratic quaternion polynomial has either two distinct isolated roots, or one double isolated root. Note also that a monic quadratic polynomial in $\mathbb{C}[t]\backslash\mathbb{R}[t]$ has only complex roots, and cannot have roots in $\mathbb{H} \setminus \mathbb{C}$ — this can be easily seen from cases 3 and 4 of [47, Theorem 2.3].

In Corollary 2.3, we gave a necessary and sufficient condition for a quaternion polynomial $\mathcal{Q}(t) \in \mathbb{H}[t] \setminus \mathbb{C}[t]$ to have a complex root $\mathbf{z}$. In this case we have also given a specific formula for its roots. The other root $\mathcal{A}$ of $\mathcal{Q}(t)$ will be necessarily complex or non–complex.

Suppose that

$$\mathcal{Q}(t) = (t - \mathcal{B})(t - \mathbf{z}).$$

By Proposition 3.1, we have the following cases:

- $\mathcal{B}$ is similar to $\mathbf{z}$. Then $\mathbf{z}$ is the only root of $\mathcal{Q}(t)$ of multiplicity 2.
- $\mathcal{B}$ is not similar to $\mathbf{z}$. Then $\mathcal{A} \neq \mathbf{z}$ and so, $\mathcal{Q}(t)$ has two distinct roots.

The form of $\mathcal{A}$ may be understood as follows. Suppose that $\mathcal{B}$ is not similar to $\mathbf{z}$. By

$$\mathcal{Q}(t) = (t - \mathcal{B})(t - \mathbf{z}) = t^2 + \mathcal{Q}_1 t + \mathcal{Q}_0$$

we obtain

$$\mathcal{Q}_1 = -(\mathcal{B} + \mathbf{z}) \quad \text{and} \quad \mathcal{Q}_0 = \mathcal{B}\,\mathbf{z}.$$

On the other hand, the roots of $\mathcal{Q}(t)$ are

$$\mathbf{z} \quad \text{and} \quad \mathcal{A} = (\mathcal{B}^* - \mathbf{z})^{-1}\mathcal{B}(\mathcal{B}^* - \mathbf{z}).$$

Substituting $\mathcal{B} = -\,\mathcal{Q}_1 - \mathbf{z}$ and $\mathbf{z} = -\mathbf{c}_2^*/\mathbf{b}_2^*$ into the latter expression, we obtain

$$\mathcal{A} = -\,(\mathcal{C}_1 + 2\,\mathrm{scal}(-\mathbf{c}_2^*/\mathbf{b}_2^*))^{-1}(\mathcal{Q}_1 - \mathbf{c}_2^*/\mathbf{b}_2^*)(\mathcal{Q}_1 + 2\,\mathrm{scal}(-\mathbf{c}_2^*/\mathbf{b}_2^*)),$$

which is also verified by Corollary 2.3. This special form of $\mathcal{A}$ holds under the assumption that $\mathcal{B}$ is not similar to $\mathbf{z}$.

In [47] Huang and So presented explicit formulas for the solutions of a quadratic quaternion equation. Below we express these formulas in terms of the real components of the coefficients of the quadratic equation.

Theorem 3.1. *Let $\mathcal{Q}(t) = t^2+\mathcal{B}t+\mathcal{C} \in \mathbb{H}[t]\backslash\mathbb{C}[t]$. Set $\mathcal{B} = b_1+b_2\mathbf{i}+b_3\mathbf{j}+b_4\mathbf{k}$ and $\mathcal{C} = c_1+c_2\mathbf{i}+c_3\mathbf{j}+c_4\mathbf{k}$, where $b_i, c_i \in \mathbb{R}$ for $i = 1, 2, 3, 4$. Then we have the following cases.*

1. *The polynomial $\mathcal{Q}(t)$ has a double root if and only if* $\mathrm{vect}(\mathcal{B}) \neq 0$*,*
$$|\mathrm{vect}\mathcal{B})||\mathrm{vect}(\mathcal{C})| - \frac{b_1}{2}|\mathrm{vect}(\mathcal{B})|^2 = 0$$
and
$$|\mathrm{vect}(\mathcal{B})|^4 + 4c_1|\mathrm{vect}(\mathcal{B})|^2 - 4|\mathcal{C}|^2 = 0.$$
Furthermore, this root is complex if and only if
$$2b_2c_3 - 2b_3c_2 - b_4|\mathrm{vect}(\mathcal{B})|^2 = 2b_4c_2 - 2b_2c_4 - b_3|\mathrm{vect}(\mathcal{B})|^2 = 0.$$
In this case, the root is not a real number.

2. *The polynomial $\mathcal{Q}(t)$ has two distinct roots, and at most one of them is complex. In this case, $\mathcal{Q}(t)$ has exactly one complex root if and only if*
$$(c_3 - c_4\mathrm{i})^2 - (c_3 - c_4\mathrm{i})(b_1 + b_2\mathrm{i})(b_3 - b_4\mathrm{i}) + (c_1 + c_2\mathrm{i})(b_3 - b_4\mathrm{i})^2 = 0.$$

Proof: 1. Substituting $\mathcal{B} = b_1+b_2\mathbf{i}+b_3\mathbf{j}+b_4\mathbf{k}$ and $\mathcal{C} = c_1+c_2\mathbf{i}+c_3\mathbf{j}+c_4\mathbf{k}$ into the conditions for a double root in [47, Corollary 2.6] and combining them, we obtain the first two conditions (a). Using [47, Theorem 2.3 (case 4)] we see that this double root is
$$x = \left(-\frac{b_2c_2 + b_3c_3 + b_4c_4}{|\mathrm{vect}(\mathcal{B})|^2}\right) + \left(-\frac{b_2}{2} + \frac{b_3c_4 - b_4c_3}{|\mathrm{vect}(\mathcal{B})|^2}\right)\mathbf{i}+$$
$$\left(-\frac{b_3}{2} + \frac{b_4c_2 - b_2c_4}{|\mathrm{vect}(\mathcal{B})|^2}\right)\mathbf{j} + \left(-\frac{b_4}{2} + \frac{b_2c_3 - b_3c_2}{|\mathrm{vect}(\mathcal{B})|^2}\right)\mathbf{k}.$$
In order to have a complex root we set the coefficients of $\mathbf{j}$ and $\mathbf{k}$ equal to zero, and we directly obtain the last two conditions in (a). In this case, the root cannot be real. Indeed, since $\mathcal{Q}(t)$ can be written as $\mathcal{Q}(t) = (t - \mathcal{A}_1)(t - \mathcal{A}_2)$ where $\mathcal{A}_2$ is the real double root and $\mathcal{A}$ a non–complex quaternion, then $\mathcal{A}_1$ must be similar to $\mathcal{A}_2$, which is a contradiction.

2. If $\mathcal{Q}(t)$ has two distinct complex roots, then Proposition 2.1 implies that $\mathcal{Q}(t)$ is a complex polynomial, which is not the case. So $\mathcal{Q}(t)$ cannot

have distinct complex roots, and consequently at most one of them can be complex. Thus, Proposition 2.1, after substituting $\mathcal{B} = (b_1+b_2\mathbf{i})+(b_3+b_4\mathbf{i})\mathbf{j}$ and $\mathcal{C} = (c_1 + c_2\mathbf{i}) + (c_3 + c_4\mathbf{i})\mathbf{j}$, gives the stated condition in (b). ∎

Remark 3.2. In Corollary 2.3 we give a necessary and sufficient condition in order for a quadratic quaternion polynomial to have a complex root. In the above Theorem we clarify this situation by giving more precise statements. So we present the necessary and sufficient conditions for a quadratic quaternion equation to have a double complex root or two distinct roots and exactly one of them being complex.

Corollary 3.1. *Let $\mathcal{Q}(t) = t^2 + \mathcal{B}t + \mathcal{C} \in \mathbb{H}[t] \setminus \mathbb{C}[t]$. Then, $\mathcal{Q}(t)$ has at least one non–complex root if and only if at least one of the four conditions in Theorem 3.1(a) does not hold.*

Proof: If at least one of the two first conditions of Theorem 3.1(a) does not hold, then the polynomial $\mathcal{Q}(t)$ does not have a double root, so it has two distinct roots. Since $\mathcal{Q}(t)$ is a non–complex polynomial, at least one of these roots is a non–complex number. In the case where at least one of the last two conditions of Theorem 3.1(a) is not valid, then the double root is not complex, and is thus a double quaternion root. Hence, if one of the four conditions of Theorem 3.1(a) does not hold, then $\mathcal{Q}(t)$ has at least one non–complex root.

Conversely, suppose that $\mathcal{Q}(t)$ has at least one non–complex root. If $\mathcal{Q}(t)$ has two distinct roots, then one of the first two conditions of Theorem 3.1(a) is not valid. Also, if $\mathcal{Q}(t)$ has a double root, then this root is not complex, so one of the last two conditions of Theorem 3.1(a) does not hold. Thus, in any case, at least one of the four conditions of Theorem 3.1(a) does not hold. ∎

3.3 Scalar-vector algorithm for the roots of quadratic quaternion polynomials

In this section we follow a different approach to computing the quaternion roots of quadratic equations, based on the scalar–vector quaternion representation. We shall apply these results in a next section to analyze certain root properties of the quadratic quaternion polynomials that generate quintic RRMF curves.

Because of widespread familiarity with the basic vector operations in $\mathbb{R}^3$ (i.e., dot and cross products), the *scalar–vector form* of quaternions provides a highly accessible approach [70] to performing computations on them.

We consider, for given quaternion coefficients $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_0$, the quadratic equation

$$\mathcal{A}_2 t^2 + \mathcal{A}_1 t + \mathcal{A}_0 = 0 \tag{3.1}$$

in the quaternion variable t, where $\mathcal{A}_0, \mathcal{A}_2 \neq 0$. There is no loss of generality in assuming the quadratic equation (3.1) to be monic. Actually, the equation (3.1) reduces to the form

$$t^2 + \mathcal{B}t + \mathcal{C} = 0 \tag{3.2}$$

through (left) multiplication with $\mathcal{A}_2^{-1}$ when $\mathcal{A}_2 \neq 0$, and when $\mathcal{A}_2 = 0$ it is linear with the trivial solution $t = -\mathcal{A}_1^{-1}\mathcal{A}_0$. We also assume that $\mathcal{A}_0 \neq 0$ in (3.1), since otherwise this equation has the trivial solutions $t = 0$ and $t = -\mathcal{A}_2^{-1}\mathcal{A}_1$

Remark 3.3. We note that, because of the non–commutative nature of the quaternion product, the familiar "completing the square" process cannot be employed to compute the roots of (3.1). In particular,

$$t^2 + \mathcal{B}\,t + \mathcal{C} \neq (t + \tfrac{1}{2}\mathcal{B})^2 - \tfrac{1}{4}\mathcal{B}^2 + \mathcal{C}\,,$$

since, in general, we have

$$(t + \tfrac{1}{2}\mathcal{B})^2 - \tfrac{1}{4}\mathcal{B}^2 = t^2 + \tfrac{1}{2}(\mathcal{B}\,t + t\,\mathcal{B}) \neq t^2 + \mathcal{B}\,t\,.$$

In the following the quaternion variable will be denoted by $\mathcal{Q} = (q, \mathbf{q})$.

3.3.1 Scalar–vector solution for roots

If we set $\mathcal{B} = (b, \mathbf{b})$, $\mathcal{C} = (c, \mathbf{c})$, and $\mathcal{Q} = (q, \mathbf{q})$, equation (3.2) may be expressed in the scalar and vectors components

$$q^2 - |\mathbf{q}|^2 + b\,q - \mathbf{b}\cdot\mathbf{q} + c = 0\,, \tag{3.3}$$

$$(2q + b)\,\mathbf{q} \,+\, q\,\mathbf{b} \,+\, \mathbf{b} \times \mathbf{q} \,+\, \mathbf{c} \,=\, \mathbf{0}\,, \tag{3.4}$$

which are equivalent with a system of four quadratic equations in the scalar part q and (the three components of) the vector part $\mathbf{q}$ of $\mathcal{Q}$.

Before studying the general solution of (3.2), we consider two special cases:

- 1st case: If $\mathcal{B}$ and $\mathcal{C}$ are both real i. e. $\mathbf{b} = \mathbf{c} = \mathbf{0}$ ((3.2) has real coefficients). In this case the solutions of (3.2) are summarized in the following Lemma.

Lemma 3.1. *When the coefficients $\mathcal{B}$ and $\mathcal{C}$ are both real, i.e., $\mathbf{b} = \mathbf{c} = \mathbf{0}$, the solutions of the quadratic equation* (3.2) *are*

- *the double real root $\mathcal{Q} = \frac{1}{2}(-b, \mathbf{0})$ when $b^2 - 4c = 0$;*
- *the two real roots $\mathcal{Q} = \frac{1}{2}(-b \pm \sqrt{b^2 - 4c}, \mathbf{0})$ when $b^2 - 4c > 0$;*
- *the "spherical root" $\mathcal{Q} = \frac{1}{2}(-b, \sqrt{4c - b^2}\,(\lambda\,\mathbf{i} + \mu\,\mathbf{j} + \nu\,\mathbf{k}))$, where λ, μ, ν are real numbers satisfying $\lambda^2 + \mu^2 + \nu^2 = 1$, when $b^2 - 4c < 0$.*

Proof: When $\mathbf{b} = \mathbf{c} = \mathbf{0}$, equations (3.3)–(3.4) reduce to

$$q^2 \,-\, |\mathbf{q}|^2 \,+\, b\,q \,+\, c \,=\, 0 \qquad \text{and} \qquad (2q + b)\,\mathbf{q} \,=\, \mathbf{0}\,.$$

From the second equation we have that $\mathbf{q} = \mathbf{0}$ or $q = -\frac{1}{2}\,b$. In the former case, the first equation reduces to $q^2 + b\,q + c = 0$, with no real roots if $b^2 - 4c < 0$; a double root $q = -\frac{1}{2}\,b$ if $b^2 - 4c = 0$; and distinct roots $q = \frac{1}{2}(-b \pm \sqrt{b^2 - 4c})$ if $b^2 - 4c > 0$. In the latter case, the first equation gives $|\mathbf{q}|^2 = c - \frac{1}{4}\,b^2$, which is satisfied by any vector of the form

$$\mathbf{q} \,=\, \tfrac{1}{2}\,\sqrt{4c - b^2}\,(\lambda\,\mathbf{i} + \mu\,\mathbf{j} + \nu\,\mathbf{k})$$

with $\lambda^2 + \mu^2 + \nu^2 = 1$ when $b^2 - 4c < 0$; by $\mathbf{q} = \mathbf{0}$ when $b^2 - 4c = 0$; and by no real vector when $b^2 - 4c > 0$. ∎

- 2nd case: If $\mathcal{B}$ and $\mathcal{C}$ are not both real. We distinguish two classes of solutions $(q, \mathbf{q})$ to the system (3.3)–(3.4), namely, solutions with (i) $q \neq -\frac{1}{2}\,b$, and we shall call them *generic* and with (ii) $q = -\frac{1}{2}\,b$, which we shall call *singular* quaternion roots of (3.1).

Generic roots

If $(q, \mathbf{q})$ is a solution of class (i) with $2q + b \neq 0$, then through the MAPLE, equation (3.4) may be solved to express $\mathbf{q}$ in terms of q, b, c and $\mathbf{b}, \mathbf{c}$ as

$$\mathbf{q} = \frac{1}{2q+b}\left[\frac{(2q+b)\,\mathbf{b}\times\mathbf{c} - (2q+b)^2\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{b}}{(2q+b)^2 + |\mathbf{b}|^2} - q\,\mathbf{b}\right]. \tag{3.5}$$

By setting $x = (2q+b)^2$ and substituting (3.5) into (3.3), further analysis using MAPLE indicates that the latter equation can be factorized to obtain

$$(x + |\mathbf{b}|^2)(x^3 + a_2x^2 + a_1x + a_0) = 0, \tag{3.6}$$

where

$$\begin{aligned} a_2 &= 2\,|\mathbf{b}|^2 - b^2 + 4c, \\ a_1 &= (|\mathbf{b}|^2 - b^2 + 4c)\,|\mathbf{b}|^2 - |b\,\mathbf{b} - 2\,\mathbf{c}|^2, \\ a_0 &= -\,(b\,|\mathbf{b}|^2 - 2\,\mathbf{b}\cdot\mathbf{c})^2. \end{aligned} \tag{3.7}$$

For a class (i) solution of equation (3.1), we must have $x = (2q+b)^2 > 0$, since $(2q+b)^2 \neq 0$. So, by (3.6) we have that the cubic

$$x^3 + a_2x^2 + a_1x + a_0 \tag{3.8}$$

must possess a positive real root. *Cardano's method* [77] offers a closed–form solution for the roots of this cubic. However, useful information about the number of its positive roots can be deduced, without actually computing them, by using the *Descartes Rule of Signs* [77].

Lemma 3.2. *If $a_0 \neq 0$, the cubic defined by* (3.7)–(3.8) *has one positive real root.*

Proof: Descartes Law of Signs states that the number of positive real roots of a polynomial is less than the number of its coefficient sign changes by an even amount. Now the cubic (3.8) is monic, and from (3.7) we have $a_0 < 0$ when $a_0 \neq 0$ (the case $a_0 = 0$ is treated in Section 3.3.1 below). Hence, the number of possible coefficient sign changes may be categorized[1] as follows:

[1]We do not explicitly address the cases $a_1 = 0$ or $a_2 = 0$, since in these instances the number of sign changes cannot exceed the indicated amounts.

(a) there is one sign change if (a_2, a_1) has signature $(+,+)$ or $(+,-)$;

(b) there are two sign changes if (a_2, a_1) has signature $(-,+)$.

We show that case (b) is impossible. From (3.7), the conditions $a_2 < 0$ and $a_1 > 0$ are equivalent to

$$c < \frac{b^2 - 2\,|\mathbf{b}|^2}{4} \qquad \text{and} \qquad c > \frac{b^2|\mathbf{b}|^2 - |\mathbf{b}|^4 + |\,b\,\mathbf{b} - 2\,\mathbf{c}\,|^2}{4\,|\mathbf{b}|^2}\,.$$

In order for these inequalities to hold, we must have

$$b^2|\mathbf{b}|^2 - 2\,|\mathbf{b}|^4 \;>\; b^2|\mathbf{b}|^2 - |\mathbf{b}|^4 + |\,b\,\mathbf{b} - 2\,\mathbf{c}\,|^2\,,$$

or, equivalently,

$$|\mathbf{b}|^4 \,+\, |\,b\,\mathbf{b} - 2\,\mathbf{c}\,|^2 \,<\, 0\,.$$

Since this is impossible, the cubic defined by (3.7)–(3.8) has one coefficient sign change, and thus one positive real root, when $a_0 \neq 0$. ∎

Let ρ be the positive root of (3.8) when $a_0 \neq 0$. Since $\rho = (2q+b)^2$, this yields two distinct values

$$q \,=\, \frac{-\,b \pm \sqrt{\rho}}{2} \tag{3.9}$$

for the scalar parts q of the roots $\mathcal{Q}$ of (3.1), with corresponding vector parts $\mathbf{q}$ specified by (3.5). Now using (3.9) we can re–write (3.5) as

$$\mathbf{q} \,=\, \frac{\mathbf{b}\times\mathbf{c}}{\rho + |\mathbf{b}|^2} \,-\, \frac{\mathbf{b}}{2} \,\pm\, \frac{1}{\sqrt{\rho}}\left[\,\frac{b\,\mathbf{b}}{2} \,-\, \frac{\rho\,\mathbf{c} + (\mathbf{b}\cdot\mathbf{c})\,\mathbf{b}}{\rho + |\mathbf{b}|^2}\,\right]\,, \tag{3.10}$$

and thus from (3.9)–(3.10) the two quaternion roots of (3.1) can be expressed as

$$\mathcal{Q} \,=\, \left(-\,\frac{b}{2}\,,\,\frac{\mathbf{b}\times\mathbf{c}}{\rho + |\mathbf{b}|^2} - \frac{\mathbf{b}}{2}\right) \,\pm\, \frac{1}{\sqrt{\rho}}\left(\frac{\rho}{2}\,,\,\frac{b\,\mathbf{b}}{2} \,-\, \frac{\rho\,\mathbf{c} + (\mathbf{b}\cdot\mathbf{c})\,\mathbf{b}}{\rho + |\mathbf{b}|^2}\right)\,, \tag{3.11}$$

where ρ is the unique positive root of (3.8) with $a_0 \neq 0$.

Remark 3.4. If $\mathbf{b}$ and $\mathbf{c}$ are linearly dependent, the roots (3.11) reduce to

$$\mathcal{Q} \,=\, -\,\frac{1}{2}\,(b,\mathbf{b}) \,\pm\, \frac{1}{2\sqrt{\rho}}\,(\rho, b\,\mathbf{b} - 2\,\mathbf{c})\,,$$

so the vector parts of both roots are also linearly dependent.

Singular roots

For class (ii) with $2q+b=0$, we have $q = -\frac{1}{2}b$ and equation (3.4) becomes

$$\mathbf{b}\times\mathbf{q} \,=\, \tfrac{1}{2}b\,\mathbf{b} \,-\, \mathbf{c}\,.$$

Now the quantities b, $\mathbf{b}$, $\mathbf{c}$ cannot be freely specified if this equation is to be satisfied. Specifically, since $\mathbf{b}\cdot(\mathbf{b}\times\mathbf{q}) = 0$, we must have

$$\tfrac{1}{2}b\,|\mathbf{b}|^2 - \mathbf{b}\cdot\mathbf{c} \,=\, 0, \tag{3.12}$$

which is equivalent to $a_0 = 0$ in (3.7), i.e., $x=0$ is a root of (3.8). Note that (3.12) is automatically satisfied when $\mathbf{b}=\mathbf{0}$. However, equation (3.4) with $2q+b=0$ and $\mathbf{b}=\mathbf{0}$ can only be satisfied when we also have $\mathbf{c}=\mathbf{0}$, which corresponds to the case of real coefficients treated in Lemma 3.1. When condition (3.12) is satisfied with $\mathbf{b}\neq\mathbf{0}$, we have

$$\mathbf{q} \,=\, \frac{\mathbf{b}\times\mathbf{c}}{|\mathbf{b}|^2} \,+\, \gamma\,\mathbf{b}\,, \tag{3.13}$$

where γ is a free parameter. Substituting (3.13) and $q = -\frac{1}{2}b$ into (3.3), and noting that $|\mathbf{b}|\neq 0$, then gives the quadratic equation

$$|\mathbf{b}|^6\gamma^2 \,+\, |\mathbf{b}|^6\gamma \,+\, |\mathbf{b}\times\mathbf{c}|^2 + \tfrac{1}{4}b^2|\mathbf{b}|^4 - c\,|\mathbf{b}|^4 \,=\, 0 \tag{3.14}$$

in γ. In order for (3.14) to have real roots, we require that

$$|\mathbf{b}|^6 - 4\,|\mathbf{b}\times\mathbf{c}|^2 - b^2|\mathbf{b}|^4 + 4\,c\,|\mathbf{b}|^4 \,\geq\, 0\,, \tag{3.15}$$

Now from (3.12) we have $b^2|\mathbf{b}|^4 = 4\,(\mathbf{b}\cdot\mathbf{c})^2$, and using the identity

$$|\mathbf{b}\times\mathbf{c}|^2 \,+\, (\mathbf{b}\cdot\mathbf{c})^2 \,=\, |\mathbf{b}|^2|\mathbf{c}|^2\,,$$

the condition (3.15) can be reduced to

$$|\mathbf{b}|^4 + 4\,c\,|\mathbf{b}|^2 - 4\,|\mathbf{c}|^2 \,\geq\, 0\,. \tag{3.16}$$

In summary, class (ii) roots exist only when conditions (3.12) and (3.16) are *both* satisfied, i.e., $a_0 = 0$ and (3.14) has a real root γ. The quaternion roots of (3.1) can then be expressed as

$$\mathcal{Q} \,=\, \left(-\frac{b}{2}\,,\, \frac{\mathbf{b}\times\mathbf{c}}{|\mathbf{b}|^2} - \frac{\mathbf{b}}{2}\right) \,\pm\, \left(0\,,\, \frac{\sqrt{|\mathbf{b}|^4 + 4\,c\,|\mathbf{b}|^2 - 4\,|\mathbf{c}|^2}}{2\,|\mathbf{b}|^2}\,\mathbf{b}\right)\,. \tag{3.17}$$

Comparing (3.11) and (3.17), we see that singular roots differ from generic roots in having identical scalar parts. If $\mathbf{b}$ and $\mathbf{c}$ are linearly dependent, both roots have vector parts linearly dependent on $\mathbf{b}$ (see Remark 3.4).

Double roots

In the generic case ($a_0 \neq 0$) the roots (3.11) are necessarily distinct, since $\rho > 0$ and hence the scalar parts differ. In the singular case ($a_0 = 0$) the roots (3.17) have coincident scalar parts, but their vector parts are usually different since equation (3.14) generically yields two distinct γ values in expression (3.13).

Clearly, equation (3.1) admits a double root only in the singular case when (3.12) is satisfied, with the further requirement that (3.14) has a double root γ, so that the vector parts (3.13) of the roots coincide, as well as the scalar parts. Now equation (3.14) has a double root when its discriminant vanishes, which means that (3.16) holds with equality. Hence, the two conditions

$$\tfrac{1}{2}\, b\, |\mathbf{b}|^2 = \mathbf{b} \cdot \mathbf{c} \qquad \text{and} \qquad |\mathbf{b}|^4 + 4\, c\, |\mathbf{b}|^2 = 4\, |\mathbf{c}|^2 \tag{3.18}$$

together specify when equation (3.1) has a double root. If these conditions are satisfied, the double root is defined by the first term on the right hand side of (3.17), and using the first condition in (3.18) it can be expressed as

$$\mathcal{Q} = \left(-\frac{\mathbf{b} \cdot \mathbf{c}}{|\mathbf{b}|^2} \,,\, \frac{\mathbf{b} \times \mathbf{c}}{|\mathbf{b}|^2} - \frac{\mathbf{b}}{2} \right) . \tag{3.19}$$

Note that, although b and c do not appear explicitly in (3.19), the double root depends on them implicitly through the satisfaction of conditions (3.18).

3.4 Algorithm & computed examples

The preceding analysis of the roots of the quadratic quaternion equation (3.1) is summarized in the following algorithm.

Algorithm

input: quaternion coefficients $\mathcal{B} = (b, \mathbf{b})$ and $\mathcal{C} = (c, \mathbf{c})$

1. if conditions (3.12) and (3.16) are both satisfied, go to step 4;
2. compute the unique positive real root ρ of the cubic (3.8);

3. compute two quaternion roots from (3.11) and go to **output**;

4. if condition (3.16) is satisfied with equality go to step 6;

5. compute two quaternion roots from (3.17) and go to **output**;

6. compute the double quaternion root from (3.19);

output: two quaternion roots $\mathcal{Q} = (q, \mathbf{q})$.

The following simple examples serve to illustrate the above algorithm.

Example 3.4.1 Consider the quadratic equation (3.1) with $\mathcal{B} = (0, \mathbf{j})$, $\mathcal{C} = (0, \mathbf{k})$. Since $b = c = 0$ and $\mathbf{b} = \mathbf{j}$, $\mathbf{c} = \mathbf{k}$ the cubic (3.8) becomes

$$x^3 + 2x^2 - 3x = 0\,,$$

with roots $x = -3, 0, 1$. Thus, from the positive root we obtain $(2q+b)^2 = 1$, and hence $q = \pm\frac{1}{2}$. Expression (3.5) then gives the corresponding vector parts as $\mathbf{q} = \frac{1}{2}(\mathbf{i} - \mathbf{j} \mp \mathbf{k})$. Hence, in this case, we have the generic right roots

$$\mathcal{Q}_1 = \tfrac{1}{2}(1, \mathbf{i} - \mathbf{j} - \mathbf{k}) \qquad \text{and} \qquad \mathcal{Q}_2 = \tfrac{1}{2}(-1, \mathbf{i} - \mathbf{j} + \mathbf{k})\,, \tag{3.20}$$

and one can easily verify that both satisfy $\mathcal{Q}^2 + (0, \mathbf{j})\,\mathcal{Q} + (0, \mathbf{k}) = 0$.

Example 3.4.2 For equation (3.1) with the coefficients

$$\mathcal{B} = \left(-2, \frac{\mathbf{j} + \mathbf{k}}{\sqrt{2}}\right) \qquad \text{and} \qquad \mathcal{C} = \left(2, \frac{\mathbf{j} - \mathbf{k}}{\sqrt{2}}\right)\,,$$

the cubic (3.8) becomes

$$x^3 + 6x^2 - 3x - 4 = 0\,,$$

with the positive root $x = 1$ and negative roots $\frac{1}{2}(-7 \pm \sqrt{33})$. The positive root gives $(2q + b)^2 = 1$, and since $b = -2$ the roots have scalar parts $q = \frac{1}{2}$ or $\frac{3}{2}$. From (3.5), the corresponding vector parts are then $\mathbf{q} = \frac{1}{2}(-\mathbf{i} + \sqrt{2}\,\mathbf{j})$ and $-\frac{1}{2}(\mathbf{i} + 2\sqrt{2}\,\mathbf{j} + \sqrt{2}\,\mathbf{k})$. Hence, we have the generic right roots

$$\mathcal{Q}_1 = \tfrac{1}{2}(1, -\mathbf{i} + \sqrt{2}\,\mathbf{j}) \qquad \text{and} \qquad \mathcal{Q}_2 = \tfrac{1}{2}(3, -\mathbf{i} - 2\sqrt{2}\,\mathbf{j} - \sqrt{2}\,\mathbf{k})$$

and one can verify that, for the given coefficients, they both satisfy (3.1).

Example 3.4.3 Consider now equation (3.1) with $\mathcal{B} = (2, \mathbf{j})$ and $\mathcal{C} = (1, \mathbf{j})$. Since $b = 2$, $c = 1$ and $\mathbf{b} = \mathbf{c} = \mathbf{j}$, the cubic (3.8) becomes

$$x^3 + 2x^2 + x = 0 ,$$

with roots $x = -1, -1, 0$. Since none of these roots is positive, there are no generic quaternion roots. For the root $x = 0$, we investigate the existence of singular roots. Since $b = 2$ and $\mathbf{b} = \mathbf{c} = \mathbf{j}$, condition (3.12) is satisfied. Equation (3.14) then becomes $\gamma^2 + \gamma = 0$, with real solutions $\gamma = -1$ and 0, for which (3.13) gives vector parts $\mathbf{q} = -\mathbf{j}$ and $\mathbf{q} = \mathbf{0}$ associated with the scalar part $q = -\frac{1}{2}b = -1$. Hence, we have the two singular right roots

$$\mathcal{Q}_1 = (-1, -\mathbf{j}) \qquad \text{and} \qquad \mathcal{Q}_2 = (-1, \mathbf{0}) ,$$

which both satisfy $\mathcal{Q}^2 + (2, \mathbf{j})\, \mathcal{Q} + (1, \mathbf{j}) = 0$.

Example 3.4.4 For equation (3.1) with $\mathcal{B} = (2, \mathbf{j})$ and $\mathcal{C} = (\frac{7}{4}, \mathbf{j} + \mathbf{k})$ the cubic (3.8) becomes

$$x^3 + 5x^2 = 0 ,$$

with roots $x = -5, 0, 0$. Since this has no positive roots, equation (3.1) has no generic quaternion roots in this case. For the singular root corresponding to $x = 0$, the scalar part is $q = -\frac{1}{2}b = -1$, and one can verify that both of the conditions (3.18) are satisfied, so this must define a double quaternion root. Equation (3.14) reduces to

$$\gamma^2 + \gamma + \tfrac{1}{4} = 0 ,$$

and has, as expected, the double root $\gamma = -\frac{1}{2}$. The corresponding vector part is then determined as $\mathbf{q} = \mathbf{i} - \frac{1}{2}\mathbf{j}$ from expression (3.13). Hence,

$$\mathcal{Q}_1 = (-1, \mathbf{i} - \tfrac{1}{2}\mathbf{j})$$

is the only quaternion root in this case, and it defines a *double* (right) root.

Chapter 4

Rational rotation–minimizing frame curves

A moving frame along a curve describes the orientation of a rigid body when it moves along its trajectory. Frames that have the unit tangent vector as one component are called *adapted* frames and among all the adapted orthonormal frames, which can be defined on a curve, we focus on the *rotation–minimizing* frame (RMF). Moreover, for practical reasons we wish to have the rational dependence on the curve parameter and thus the *Euler–Rodrigues* frame and the *rational* RMF (RRMF) are introduced. The search for curves with rational adapted frames is restricted to the particular class of polynomial curves with a special structure–which they called *Pythagorean–hodograph (PH) curves*–since only PH curves have rational unit tangents. This chapter is a brief overview of the basic theory related to adapted frames and to PH curves.

4.1 Spatial Pythagorean–hodograph curves

We shall begin this section by giving some basic definitions. Recall that a *polynomial* space curve defined by $x(t), y(t), z(t) \in \mathbb{R}[t]$ is the set

$$C = \{(x(t), y(t), z(t)) \in \mathbb{R}^3 \,|\, t \in \mathbb{R}\}.$$

We denote by $\mathbf{r}$ the parametrization of C, i. e. the map is defined by

$$t \mapsto (x(t), y(t), z(t)).$$

In the following we shall refer to the polynomial space curve C by giving its parametrization $\mathbf{r}(t)$. Here we focus on a special class of polynomial space curves which are of great importance in practical applications.

For a given space curve $\mathbf{r}(t) = (x(t), y(t), z(t))$ the *hodograph* is its parametric derivative $\mathbf{r}'(t) = (x'(t), y'(t), z'(t))$ regarded as a curve in its own right. The curve $\mathbf{r}(t)$ is said to have a *Pythagorean–hodograph* [20] if there exists a real polynomial $\sigma(t)$ such that

$$x'^2(t) + y'^2(t) + z'^2(t) \;=\; \sigma^2(t). \tag{4.1}$$

In the following by a "*Pythagorean–hodograph curve*" (*PH curve*) we shall mean any polynomial curve whose derivative is of the form (4.1).

By [8] and [13], the necessary and sufficient condition for the satisfaction of (4.1) is that the real polynomials $x'(t), y'(t), z'(t)$ must be expressible in terms of other polynomials $u(t), v(t), p(t), q(t) \in \mathbb{R}[t]$ in the form

$$\begin{aligned} x'(t) &= u^2(t) + v^2(t) - p^2(t) - q^2(t), \\ y'(t) &= 2\,[\,u(t)q(t) + v(t)p(t)\,], \\ z'(t) &= 2\,[\,v(t)q(t) - u(t)p(t)\,]. \end{aligned} \tag{4.2}$$

The polynomial

$$\sigma(t) \;=\; u^2(t) + v^2(t) + p^2(t) + q^2(t) \tag{4.3}$$

defines the *parametric speed* of the curve $\mathbf{r}(t)$, i.e., the rate of change $\mathrm{d}s/\mathrm{d}t$ of its arc length s with respect to the curve parameter t. Note that form (4.2) can be written in several different ways, corresponding to permutations of $x'(t), y'(t), z'(t)$ and $u(t), v(t), p(t), q(t)$. We shall say that the curve $\mathbf{r}(t)$ is called *regular* if $|\mathbf{r}'(t)|(t) \neq 0$, for all t. The fact that $\sigma(t)$ is a *polynomial* function (rather than the square–root of a polynomial) in t is the source of the advantageous properties of these curves.

If s=max[deg($u(t)$), deg($v(t)$), deg($p(t)$), deg($q(t)$)], i.e., if the polynomials $u(t), v(t), p(t), q(t)$ are of degree s at most, then the PH curves obtained by integrating the hodograph $\mathbf{r}'(t)$ is of *degree* $n = 2s+1$. Hence, PH curves are necessary of odd degree and we shall call them *cubic*, *quintic* and *of 7 degree* if $s = 1$, $s = 2$ and $s = 3$, respectively.

A *primitive* hodograph $\mathbf{r}'(t)$ is characterized by the fact that gcd$((x'(t)$, $y'(t)$, $z'(t)) = 1$. Otherwise is called *non–primitive*. Primitive hodographs are desirable in practice since at a common real root of $x'(t), y'(t), z'(t)$

may incur a *cusp* i.e., irregular curve points where the hodograph vanishes, or inflection points. This is why we consider polynomials $u(t)$, $v(t)$, $p(t)$, $q(t)$ having $\gcd(u(t), v(t), p(t), q(t)) = 1$, since common real roots of these polynomials incur cusps on the curve. In this work we always consider this case. However, we can see in [20] that $\gcd(u, v, p, q) = 1$ does not ensure that the hodograph is primitive. The hodograph components may have common quadratic factors with complex conjugate roots even if $\gcd(u, v, p, q) = 1$. In this case the hodograph $\mathbf{r}'(t)$ is non–primitive but the PH curve is regular i.e., $|\mathbf{r}'(t)| \neq 0$, for all real t. We shall study the case of non–primitive hodographs in Chapter 7.

In [8] Choi et al. introduced two equivalent characterization of solutions to condition (4.1) using the algebra of *quaternions* and the *Hopf map* which are greatly useful in the research of spatial PH curves.

The *quaternion formulation* provides a very elegant and concise description of this structure which contributes to the development of basic algorithms concerned with their construction, analysis, and applications of PH curves.

If

$$\mathcal{A}(t) = u(t) + \mathbf{i}\, v(t) + \mathbf{j}\, p(t) + \mathbf{k}\, q(t) \tag{4.4}$$

is a quaternion polynomial, then the product

$$\begin{aligned} \mathbf{r}'(t) \;=\; \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t) \;=\; & \left[\, u^2(t) + v^2(t) - p^2(t) - q^2(t) \,\right] \mathbf{i} \\ & + 2 \left[\, u(t)q(t) + v(t)p(t) \,\right] \mathbf{j} \\ & + 2 \left[\, v(t)q(t) - u(t)p(t) \,\right] \mathbf{k} \end{aligned} \tag{4.5}$$

generates a PH curve in $\mathbb{R}^3$ ($\mathbf{j}$ or $\mathbf{k}$ can be interposed between $\mathcal{A}(t)$, $\mathcal{A}^*(t)$ in place of $\mathbf{i}$, yielding a permutation of $u(t), v(t), p(t), q(t)$. We may express (4.5) as

$$\mathbf{r}'(t) = |\mathcal{A}(t)|^2 \mathcal{U}(t)\,\mathbf{i}\; \mathcal{U}^*(t),$$

where $\mathcal{U}(t) = \cos(\theta(t)/2) + \mathbf{n}(t)\sin(\theta(t)/2)$ is a unit quaternion expressed in terms of an angle $\theta(t)$ and a unit vector $\mathbf{n}(t)$ [20]. The product $\mathcal{U}(t)\,\mathbf{i}\; \mathcal{U}^*(t)$ defines a spatial rotation of the vector $\mathbf{i}$ by angle $\theta(t)$ about the axis vector $\mathbf{n}(t)$, while the factor $|\mathcal{A}(t)|^2$ imposes a scaling of this rotated vector $\mathbf{i}$. Using the quaternion representation of $\mathbf{r}'(t)$, we may sometimes express the quaternion polynomial $\mathcal{A}(t)$ in the Bernstein form

$$\mathcal{A}(t) = \sum_{r=0}^{s} \mathcal{A}_r \binom{s}{r} (1-t)^{s-r} t^r, \tag{4.6}$$

where s is the maximum of the degrees of $u(t)$, $v(t)$, $p(t)$, $q(t)$.

As an alternative to the quaternion representation, the *Hopf map form* generates a Pythagorean hodograph from two complex polynomials

$$\boldsymbol{\alpha}(t) \;=\; u(t) + \mathrm{i}\, v(t)\,, \quad \boldsymbol{\beta}(t) \;=\; q(t) + \mathrm{i}\, p(t) \tag{4.7}$$

through the expression

$$\mathbf{r}'(t) \;=\; H(\mathbf{a}(t), \mathbf{b}(t)) \;=\; (|\boldsymbol{\alpha}(t)|^2 - |\boldsymbol{\beta}(t)|^2, 2\,\mathrm{Re}(\boldsymbol{\alpha}(t)\overline{\boldsymbol{\beta}}(t)), 2\,\mathrm{Im}(\boldsymbol{\alpha}(t)\overline{\boldsymbol{\beta}}(t))) \tag{4.8}$$

where $H : \mathbb{C} \times \mathbb{C} \mapsto \mathbb{R}^3$ is the Hopf map [20]. The parametric speed in this case is

$$\sigma(t) = |\boldsymbol{\alpha}(t)|^2 + |\boldsymbol{\beta}(t)|^2 = u^2(t) + v^2(t) + p^2(t) + q^2(t).$$

Note that the hodograph (4.8) is primitive if $\gcd(\mathbf{a}(t), \mathbf{b}(t)) = 1$. Also note that for a PH curve of degree $n = 2s + 1$ we may assume the polynomials $\mathbf{a}(t), \mathbf{b}(t)$ in Bernstein form as

$$\mathbf{a}(t) = \sum_{r=0}^{s} \mathbf{a}_r \binom{s}{r} (1-t)^{s-r} t^r, \qquad \mathbf{b}(t) = \sum_{r=0}^{s} \mathbf{b}_r \binom{s}{r} (1-t)^{s-r} t^r.$$

The equivalence of (4.5) and (4.8) may be seen by setting $\mathcal{A}(t) = \boldsymbol{\alpha}(t) + \mathbf{k}\,\boldsymbol{\beta}(t)$, and identifying the imaginary unit i with the quaternion element $\mathbf{i}$. See [20] for a more thorough treatment of these two representations.

Remark 4.1. By (4.5) we see that the hodograph $\mathbf{r}'(t)$ of a PH curve is generated (or defined) by the quaternion polynomial $\mathcal{A}(t)$. Thus, the family of the curves $\mathbf{r}(t)$ can be obtained by integrating (4.5). For simplicity reasons, in the following we shall sometimes say that the curve $\mathbf{r}(t)$ is generated (or is defined) by the polynomial $\mathcal{A}(t)$.

Remark 4.2. Recall that -as it is mentioned- throughout this work we will always consider that $\gcd(u(t), v(t), p(t), q(t)) = 1$, unless otherwise stated.

In case where the quaternion polynomial $\mathcal{A}(t)$, given by (4.4) which generates the hodograph of a PH curve, has $\gcd(u(t), v(t), p(t), q(t)) = 1$ will be called *primitive*. Otherwise, $\mathcal{A}(t)$ is said to be *non-primitive*. Note that, by Theorem 1.6, primitive quaternion polynomials have only isolated roots.

4.2 Adapted frames on space curves

An *adapted* frame $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ on a space curve $\mathbf{r}(t)$ is an orthonormal basis defined at each curve point, where $\mathbf{f}_1$ coincides with the curve tangent $\mathbf{t} = \mathbf{r}'/|\mathbf{r}'|$ and $\mathbf{f}_2, \mathbf{f}_3$ span the normal plane, such that $\mathbf{f}_1 \times \mathbf{f}_2 = \mathbf{f}_3$. The variation of such a frame may be specified by its angular velocity

$$\boldsymbol{\omega} = \omega_1 \mathbf{f}_1 + \omega_2 \mathbf{f}_2 + \omega_3 \mathbf{f}_3$$

through the differential relations

$$\mathbf{f}_1' = \sigma\boldsymbol{\omega} \times \mathbf{f}_1 \,, \qquad \mathbf{f}_2' = \sigma\boldsymbol{\omega} \times \mathbf{f}_2 \,, \qquad \mathbf{f}_3' = \sigma\boldsymbol{\omega} \times \mathbf{f}_3 \,, \tag{4.9}$$

where $\sigma(t) = |\mathbf{r}'(t)|$ is the parametric speed of $\mathbf{r}(t)$.

The magnitude and direction of the angular velocity define the instantaneous angular speed $|\boldsymbol{\omega}|$ and the rotation axis $\boldsymbol{\omega}/|\boldsymbol{\omega}|$ of the adapted frame $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$. Now, if we choose a particular adapted frame $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ reference, then for any scalar function $\theta(t)$ we can define another adapted frame [5] by

$$(\mathbf{f}_1, \cos\theta(t)\,\mathbf{f}_2 + \sin\theta(t)\,\mathbf{f}_3, -\sin\theta(t)\,\mathbf{f}_2 + \cos\theta(t)\,\mathbf{f}_3).$$

This frame corresponds to a rotation of the normal–plane vectors $\mathbf{f}_2, \mathbf{f}_3$ of the reference frame, through $\theta(t)$ at each point of the curve. Therefore, there are infinitely many adapted frames on a given space curve $\mathbf{r}(t)$.

4.2.1 Frenet adapted frame

The most ordinary adapted frame is the *Frenet frame* $(\mathbf{t}, \mathbf{h}, \mathbf{b})$ which is defined by

$$\mathbf{t} = \frac{\mathbf{r}'}{|\mathbf{r}'|} \,, \quad \mathbf{h} = \frac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}' \times \mathbf{r}''|} \times \mathbf{t} \,, \quad \mathbf{b} = \frac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}' \times \mathbf{r}''|} \,,$$

and describes [1] the *intrinsic geometry* of $\mathbf{r}(t)$. At each point of the curve the *principal normal* vector $\mathbf{h}$ points toward the center of curvature and the *binormal* $\mathbf{b} = \mathbf{t} \times \mathbf{h}$ completes the frame. As it is known, the three orthogonal planes which spanned by the vectors $(\mathbf{t}, \mathbf{h})$, $(\mathbf{h}, \mathbf{b})$ and $(\mathbf{b}, \mathbf{t})$ are called *osculating*, *normal* and *rectifying* planes at each point of $\mathbf{r}(t)$, respectively. The angular velocity of the Frenet frame is given by the *Darboux* vector

$$\mathbf{d} = \tau\,\mathbf{t} + \kappa\,\mathbf{b},$$

where κ and τ are the *curvature* and the *torsion* of the curve which are given by

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}, \quad \tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}.$$

The variation of the Frenet frame along the curve is given by the well–known [1] *Frenet–Serret equations* by considering $(\mathbf{f}_2, \mathbf{f}_3) = (\mathbf{h}, \mathbf{b})$ and use relations (4.9). If we introduce the vector $\mathbf{d}$ in Frenet–Serret equations then we have a more compact form for the derivatives $\mathbf{t}', \mathbf{h}', \mathbf{b}'$ which are

$$\mathbf{t}' = \sigma\mathbf{d} \times \mathbf{t}, \qquad \mathbf{h}' = \sigma\mathbf{d} \times \mathbf{h}, \qquad \mathbf{b}' = \sigma\mathbf{d} \times \mathbf{b},$$

From the above relations we see that the rate of change of the Frenet frame is the instantaneous rotation about the vector $\mathbf{d}$ with angular speed $|\mathbf{d}|$ [20]. Although the Frenet frame is a common choice to describe a general spatial motion of a rigid body, it is often not suitable for applications, since its normal $\mathbf{h}$ and binormal $\mathbf{b}$ vectors may appear to execute a rotation about the tangent vector $\mathbf{t}$ which is not desirable for the study of space motions. Moreover, using the Frenet frame a problem which arises is from the fact that it is not defined at points where the curvature κ vanishes (i.e., at inflection points). Even though this problem can be overcome, a serious care is required to avoid sudden reversals of the frame vectors at inflection points. This disadvantage of the Frenet frame and the need to have a frame which does not execute extra rotations on the normal plane, lead us to search for frames with minimum amount of rotations along the curve, the so–called *rotation–minimizing (adapted) frames* [44, 58].

4.2.2 Rotation–minimizing adapted frames

As previously noted, there are many adapted orthonormal frames associated with a given space curve $\mathbf{r}(t)$. Among them the *rotation–minimizing frames* (RMFs) are the ones which are of great interest in applications. If $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ is an adapted frame, the characteristic property in order for this frame to be RMF is that its angular velocity satisfies $\boldsymbol{\omega} \cdot \mathbf{f}_1 \equiv 0$, i.e., $\boldsymbol{\omega}$ has no component along $\mathbf{f}_1$. This is equivalent to

$$\mathbf{f}_3\,\mathbf{f}_2' \;=\; \mathbf{f}_3'\,\mathbf{f}_2 = 0$$

which is the necessary and sufficient condition for the frame to be RMF. In other words $\mathbf{f}_2, \mathbf{f}_3$ have no instantaneous rotation about $\mathbf{f}_1$ or equivalently, their derivatives $\mathbf{f}_2', \mathbf{f}_3'$ are always parallel to $\mathbf{f}_1$. This geometrically means that the two normal vectors $\mathbf{f}_2, \mathbf{f}_3$ rotate as little as possible around $\mathbf{f}_1$ and thus RMFs minimize the amount of rotation along the curve. Having the property of minimum twist makes RMFs very attractive in computer graphics, swept surface constructions, motion design and other similar applications [24, 25, 55, 58, 74, 80, 81]. The Frenet frame $(\mathbf{t}, \mathbf{h}, \mathbf{b})$ is not necessarily rotation–minimizing as it is shown by its angular velocity $\mathbf{d}$, since it contains the component $\tau\mathbf{t}$, which is generally non–zero for a spatial curve. In relation to the Frenet frame, the RMF can be obtained from the vectors $\mathbf{h}, \mathbf{b}$ through a rotation in the normal–plane:

$$\begin{bmatrix} \mathbf{f}_2(t) \\ \mathbf{f}_3(t) \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \mathbf{h}(t) \\ \mathbf{b}(t) \end{bmatrix} \tag{4.10}$$

and $\mathbf{f}_1 = \mathbf{t}$. The angle $\phi = \phi(t)$ specifies the difference of the two frames and it has the form [44]

$$\phi(t) \;=\; \phi_0 - \int_0^t \tau(u)\sigma(u)\,\mathrm{d}u, \tag{4.11}$$

where ϕ_0 is an arbitrary integration constant. Thus, due to the different values for ϕ_0, there are infinitely many RMFs on a given space curve which they differ from each other by a fixed angular displacement in the normal plane at each point of curve. Obviously, the angular velocity of an RMF is $\omega = \kappa\,\mathbf{b}$ since in the expression of Darboux vector $\mathbf{d}$ the term $\tau\,\mathbf{t}$ is omitted.

Since the integral (4.11) does not admit a closed–form reduction for "ordinary" polynomial or rational curves, a number of schemes have been proposed to *approximate* the rotation minimizing frame of a given curve or to approximate a curve by simpler segments with known rotation–minimizing frames. More results concerning the construction, applications and rational approximations of curves with RMF are included in [21, 25, 32, 55, 24, 53, 54, 80]. However, for PH curves the integral (4.11) is a *rational function* and thus admits closed–form integration [18]. So for any spatial PH curve exact RMFs can be computed, but in general they incur transcendental functions. Due to the preference of rational forms in computer aided design applications, since they admit exact and efficient computation, there has been interest in constructing polynomial curves with *rational rotation minimizing frame (RRMF)*–the so–called *RRMF curves.*

4.3 Rational rotation–minimizing frames

As it is mentioned, great interest has recently emerged in identifying and constructing curves that have *rational* RMFs . Such curves must be PH curves, since only PH curves have rational unit tangents. The construction of RRMF curves is thus essentially a matter of identifying constraints on the coefficients of PH curves that are sufficient and necessary for a rational RMF.

Therefore, our focus here is on the special class of PH curves with exact *rational* rotation–minimizing frames (*RRMFs*), or *RRMF curves.*

4.3.1 Euler–Rodrigues frame

As we observed, the Frenet frame is not a good reference for identifying rational RMFs because it is not rational and can exhibit singular behavior at inflection points. In order to remedy these problems, in [7] a special adapted frame has been introduced which is defined particularly on PH curves, the so–called *Euler–Rodrigues frame (ERF).*

The following definition shows the form of the ERF vectors.

Definition 4.1 The *Euler–Rodrigues frame* (ERF) on the PH curve specified by (4.4)–(4.5) is the set of orthonormal vectors defined by

$$(\mathbf{e}_1(t), \mathbf{e}_2(t), \mathbf{e}_3(t)) = \frac{(\mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t), \mathcal{A}(t)\,\mathbf{j}\,\mathcal{A}^*(t), \mathcal{A}(t)\,\mathbf{k}\,\mathcal{A}^*(t))}{|\mathcal{A}(t)|^2}. \tag{4.12}$$

Note that $\mathbf{e}_1$ is the curve tangent, while $\mathbf{e}_2, \mathbf{e}_3$ span the curve normal plane.

The ERF is not a geometrically intrinsic frame (it depends on the chosen coordinate system) but it is rational by construction and always nonsingular at inflection points. In [7] conditions were investigated under which the ERF of a PH curve can be an RMF and it was proved that:

1. For PH cubics the ERF and the Frenet frame are the same
2. PH quintics which have rotation minimizing ERF are planar curves
3. Spatial PH curves for which the ERF is RMF are at least of degree 7.

The ERF $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is given in terms of the components $u(t), v(t), p(t), q(t)$ of $\mathcal{A}(t)$ by

$$\begin{aligned}
\mathbf{e}_1 &= \frac{(u^2+v^2-p^2-q^2)\,\mathbf{i} \,+\, 2(uq+vp)\,\mathbf{j} \,+\, 2(vq-up)\,\mathbf{k}}{u^2+v^2+p^2+q^2}, \\
\mathbf{e}_2 &= \frac{2(vp-uq)\,\mathbf{i} \,+\, (u^2-v^2+p^2-q^2)\,\mathbf{j} \,+\, 2(uv+pq)\,\mathbf{k}}{u^2+v^2+p^2+q^2}, \\
\mathbf{e}_3 &= \frac{2(up+vq)\,\mathbf{i} \,+\, 2(pq-uv)\,\mathbf{j} \,+\, (u^2-v^2-p^2+q^2)\,\mathbf{k}}{u^2+v^2+p^2+q^2}.
\end{aligned} \tag{4.13}$$

Now if the PH curve defined by (4.4)–(4.5) admits a rational RMF $(\mathbf{f}_1(t),\, \mathbf{f}_2(t), \mathbf{f}_3(t))$, then $\mathbf{e}_1 = \mathbf{f}_1$ is the curve tangent, and the normal–plane vectors $\mathbf{f}_2(t), \mathbf{f}_3(t)$ must be obtainable from the ERF normal–plane vectors $\mathbf{e}_2(t), \mathbf{e}_3(t)$ by a rational rotation — i.e., for relatively prime polynomials $a(t), b(t)$ we must have

$$\begin{bmatrix} \mathbf{f}_2(t) \\ \mathbf{f}_3(t) \end{bmatrix} = \frac{1}{a^2(t)+b^2(t)} \begin{bmatrix} a^2(t)-b^2(t) & -2\,a(t)b(t) \\ 2\,a(t)b(t) & a^2(t)-b^2(t) \end{bmatrix} \begin{bmatrix} \mathbf{e}_2(t) \\ \mathbf{e}_3(t) \end{bmatrix} \tag{4.14}$$

The importance of the ERF for identifying RRMF curves may be phrased as follows.

Theorem 4.1. [45] *The PH curve defined by* (4.4)–(4.5) *has a rational RMF if and only if relatively prime polynomials* $a(t)$, $b(t)$ *exist, such that*

$$\frac{uv' - u'v - pq' + p'q}{u^2 + v^2 + p^2 + q^2} = \frac{ab' - a'b}{a^2 + b^2}. \tag{4.15}$$

Note that the expression on the left in (4.15) is the component $\omega_1 = \boldsymbol{\omega} \cdot \mathbf{t}$ of the ERF angular velocity $\boldsymbol{\omega}$ in the direction of $\mathbf{e}_1 = \mathbf{f}_1$, while that on the right is the angular velocity of the normal–plane rotation (4.14) that maps $\mathbf{e}_2, \mathbf{e}_3$ onto $\mathbf{f}_2, \mathbf{f}_3$. Thus, condition (4.15) requires the existence of a rational normal–plane rotation that exactly cancels the ω_1 component of the ERF angular velocity.

The difference in behavior of Frenet, Euler-Rodrigues and rotation-minimizing frames is visualized in Fig. 4.1.

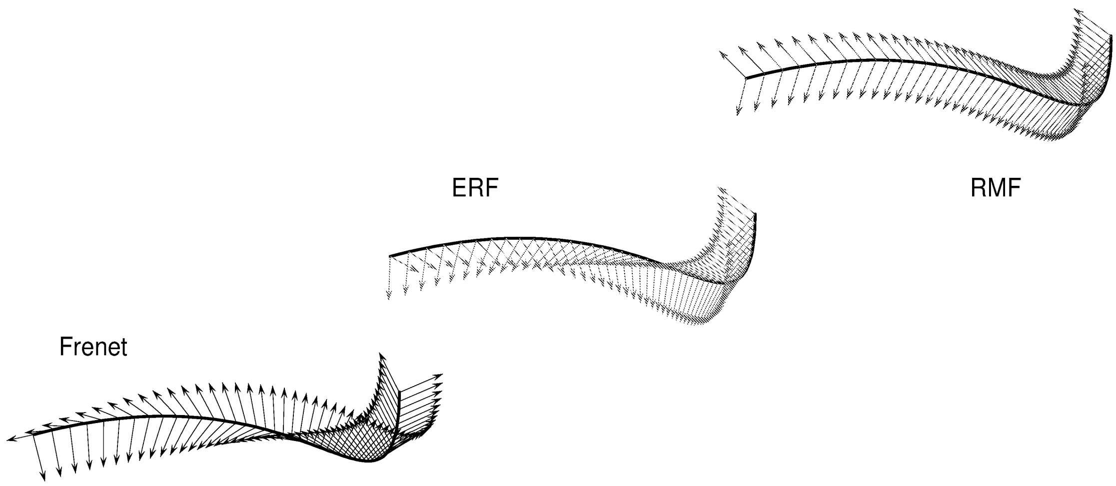

Figure 4.1: The Frenet frame (left), Euler–Rodrigues frame (center) and the rotation–minimizing frame (right). The unit tangent vector is common to all frames and it is not shown here.

Remark 4.3. Note that the numerator and denominator of the expression on the left of (4.15) can be shortly expressed in terms of $\mathcal{A}(t)$ as $\mathrm{scal}(\mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}'^*(t))$ and $|\mathcal{A}(t)|^2$, respectively.

As in [21], it is convenient to introduce the notations

$$\frac{uv' - u'v - pq' + p'q}{u^2 + v^2 + p^2 + q^2} = [u, v, p, q] = [\mathcal{A}] \quad \text{and} \quad \frac{ab' - a'b}{a^2 + b^2} = [a, b].$$

An analysis of the structure of $[u, v, p, q]$ and $[a, b]$ is presented in [21] and it is important for identifying RRMF curves. The Lemma below includes some useful properties of these quotients, which are used in next chapters.

Lemma 4.1. Let $a(t), b(t), c(t), d(t), e(t), f(t), g(t), h(t) \in \mathbb{R}[t]$, $\mathcal{C} \in \mathbb{H}$, and $\mathbf{r} = \alpha + \mathbf{i}\,\beta \in \mathbb{C}$ with $\beta \neq 0$. Then the following results hold.

1. Condition (4.15) remains unchanged if $\mathcal{A}$ is replaced by $\mathcal{C}\mathcal{A}(t)$ for any $\mathcal{C} \neq 0$.

2. $[E, F, D, H] = [e, f, d, h] \pm [c, d]$, where

$$E + F\,\mathbf{i} + G\,\mathbf{j} + H\,\mathbf{k} = (e + f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k})(c \pm \,\mathbf{i}\,d).$$

   In particular, $[C, D] = [e, f] \pm [c, d]$, where $C + \mathbf{i}\,D = (e + \mathbf{i}\,f)(c \pm \,\mathbf{i}\,d)$.

3. If $a + \mathbf{i}\,b, c + \mathbf{i}\,d$ are primitive with $[a, b] = [c, d]$ then $a + \mathbf{i}\,b = \mathbf{z}\,(c + \mathbf{i}\,d)$ for $\mathbf{z} \in \mathbb{C}$.

Proof: See [35, Lemma 2.1].

In terms of the Hopf map representation (4.7)–(4.8), the RRMF condition (4.15) is equivalent to requiring the existence of a complex polynomial $\mathbf{w}(t) = a(t) + \mathrm{i}\,b(t)$, with $\gcd(a(t), b(t)) = 1$, such that

$$\frac{\operatorname{Im}(\overline{\boldsymbol{\alpha}}\boldsymbol{\alpha}' + \overline{\boldsymbol{\beta}}\boldsymbol{\beta}')}{|\boldsymbol{\alpha}|^2 + |\boldsymbol{\beta}|^2} = \frac{\operatorname{Im}(\overline{\mathbf{w}}\mathbf{w}')}{|\mathbf{w}|^2}\,. \tag{4.16}$$

Han [45] showed that rational RMFs cannot exist on PH cubics, except on planar or on PH curves with non–primitive hodographs.

Remark 4.4. When $\mathbf{w}(t)$ is either a real polynomial or a constant, the angle $\theta(t)$ between the ERF and RMF is constant. This is equivalent to

$$\operatorname{Im}(\overline{\boldsymbol{\alpha}}\boldsymbol{\alpha}' + \overline{\boldsymbol{\beta}}\boldsymbol{\beta}') = 0 \tag{4.17}$$

Since in the computation of the RMF appears an integration constant, we may consider (4.17) as the condition identifying coincidence of the RMF and ERF (ERF=RMF). Futher analysis of this condition was presented in [7]. Note that in view of (4.15) condition (4.17) is equivalent to

$$\operatorname{scal}(\mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}'^{*}(t)) = 0. \tag{4.18}$$

Now in order to classify and characterize the RRMF curves we give the following definition.

Definition 4.2 Let $\mathcal{A}(t) = u(t) + \mathbf{i}\,v(t) + \mathbf{j}\,p(t) + \mathbf{k}\,q(t)$, $a(t) + \mathbf{i}\,b(t)$ be primitive polynomials of degrees n, m respectively, that satisfy (4.15). Then, the PH/RRMF curve $\mathbf{r}(t)$, whose hodograph is $\mathbf{r}'(t) = \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t)$, will be called of *type* (n, m) curve. Also, we say that the quaternion polynomial $\mathcal{A}(t)$ has (is of) type (n, m).

Note that in the following chapters we are specially interested in RRMF curves of type $(n, 0)$ which means that the ERF of the curve is RMF, at each curve point.

Remark 4.5. By Lemma 4.1(3), we deduce that the degrees of polynomials $a(t), b(t)$ are uniquely determined. Thus the notion of the *type* is well defined.

By [21, 23], it is known that the simplest non–planar curves with rational RMFs are quintics defined by a quadratic quaternion polynomial

$$\mathcal{A}(t) = \mathcal{A}_0(1-t)^2 + \mathcal{A}_1 2(1-t)t + \mathcal{A}_2 t^2 \tag{4.19}$$

in (4.5), or quadratic complex polynomials

$$\boldsymbol{\alpha}(t) = \boldsymbol{\alpha}_0(1-t)^2+\boldsymbol{\alpha}_1 2(1-t)t+\boldsymbol{\alpha}_2 t^2, \quad \boldsymbol{\beta}(t) = \boldsymbol{\beta}_0(1-t)^2+\boldsymbol{\beta}_1 2(1-t)t+\boldsymbol{\beta}_2 t^2. \tag{4.20}$$

in (4.8). The following result characterizes the quintic RRMF curves of type $(2, 2)$.

Theorem 4.2. *The PH curve defined by* (4.19) *with* $\mathcal{A}_0 = 1$ *satisfies* (4.15) *with* $a(t), b(t)$ *quadratic if and only if the coefficients of* $\mathcal{A}(t)$ *satisfy the constraint*

$$\text{vect}(\mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_0^*) \;=\; \mathcal{A}_1\,\mathbf{i}\,\mathcal{A}_1^*. \tag{4.21}$$

Equivalently, the PH curve defined by (4.20) *satisfies* (4.16) *with* $a(t), b(t)$ *quadratic if and only if the coefficients of* $\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)$ *satisfy the constraints*

$$\mathit{Re}(\boldsymbol{\alpha}_0\overline{\boldsymbol{\alpha}}_2 - \boldsymbol{\beta}_0\overline{\boldsymbol{\beta}}_2) \;=\; |\boldsymbol{\alpha}_1|^2 - |\boldsymbol{\beta}_1|^2 \qquad \textit{and} \qquad \boldsymbol{\alpha}_0\overline{\boldsymbol{\beta}}_2 + \boldsymbol{\alpha}_2\overline{\boldsymbol{\beta}}_0 \;=\; 2\,\boldsymbol{\alpha}_1\overline{\boldsymbol{\beta}}_1. \tag{4.22}$$

Proof: See [21, Propositions 3 and 4].

An algorithm to construct rational rotation–minimizing motions — interpolating initial/final positions and orientations of a rigid body — was presented in [24], using the RRMF quintics of type $(2,2)$ identified by this theorem.

The RRMF curves of type $(2,1)$ have a more intricate algebraic structure, with no simple characterization by coefficient constraints such as (4.21) or (4.22).

4.4 Reduction to normal form

In this section we present a reduction that allows us to simplify the study of RRMF curves. Recall that by [21], the analysis of spatial PH curves can be simplified by an appropriate scaling/rotation transformation which eliminates non-essential freedoms that do not influence the intrinsic nature of the curves. We call this transformation *reduction to normal form.*

When $\boldsymbol{\alpha}(t) = u(t) + \mathbf{i}\,v(t)$, $\boldsymbol{\beta}(t) = q(t) + \mathbf{i}\,p(t)$ define a PH curve with hodograph $\mathbf{r}'(t)$ specified by (4.5), the map

$$\begin{bmatrix} \boldsymbol{\alpha}(t) \\ \boldsymbol{\beta}(t) \end{bmatrix} \to \begin{bmatrix} \boldsymbol{\mu}_1 & -\overline{\boldsymbol{\mu}}_2 \\ \boldsymbol{\mu}_2 & \overline{\boldsymbol{\mu}}_1 \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha}(t) \\ \boldsymbol{\beta}(t) \end{bmatrix} \tag{4.23}$$

defines a scaling/rotation of the hodograph in $\mathbb{R}^3$ which does not change its intrinsic nature [21].

The following Lemma shows that transformation (4.23) does not influence the RRMF property of a spatial PH curve.

Lemma 4.2. *If the RRMF condition (4.16) is satisfied by complex polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ and $\mathbf{w}(t)$, it is also satisfied upon replacing them by $\boldsymbol{\mu}_1\,\boldsymbol{\alpha}(t) - \overline{\boldsymbol{\mu}}_2\,\boldsymbol{\beta}(t)$, $\boldsymbol{\mu}_2\,\boldsymbol{\alpha}(t) + \overline{\boldsymbol{\mu}}_1\,\boldsymbol{\beta}(t)$ and $\boldsymbol{\eta}\,\mathbf{w}(t)$, for any complex numbers $(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \neq (0,0)$ and $\boldsymbol{\eta} \neq 0$.*

Proof: See [34, Lemma 2].

By the following lemma, we may consider, without loss of generality that $u(t) = t^n + \ldots + u_1(t) + u_0$ and $v(t), p(t), q(t)$ are of degree $m-1$ at most.

Lemma 4.3. *Let $\boldsymbol{\alpha}(t) = u(t) + \mathbf{i}\, v(t)$, $\boldsymbol{\beta}(t) = q(t) + \mathbf{i}\, p(t)$ be complex polynomials, where $u(t), v(t), p(t), q(t) \in \mathbb{R}[t]$ of degree $m \geq 1$. Then, complex values $\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_2$ can be chosen such that, under the transformation (4.23) the polynomials $v(t), p(t), q(t)$ are of degree $m-1$ at most.*

Proof: See [34, Lemma 1].

We call the quadruple of polynomials $(u(t), v(t), p(t), q(t))$ of this form *normal.*

Planar and straight line PH curves

Since every straight line and every planar PH curve is trivially an RRMF curve, and we are interested in true space curves, when (4.5) or (4.8) define straight line or planar curves we shall say that define *degenerate* spatial PH curves. In [34] are presented the necessary and sufficient conditions under which a spatial PH curve is a degenerate curve.

Proposition 4.1. *Let a quaternion polynomial $\mathcal{A}(t) = u(t) + \mathbf{i}\, v(t) + \mathbf{j}\, p(t) + \mathbf{k}\, q(t)$ defined by the normal quadruple $(u(t), v(t), p(t), q(t))$ and $\mathbf{r}(t)$ be a PH curve with hodograph $\mathbf{r}'(t) = \mathcal{A}(t)\, \mathbf{i}\, \mathcal{A}^*(t)$. Then*

1. *$\mathbf{r}(t)$ is a planar curve, other than a straight line, if and only if*

$$(p^2 + q^2)\,(uv' - u'v) \;+\; (u^2 + v^2)\,(pq' - p'q) = 0 \tag{4.24}$$

   with $(p(t), q(t)) \neq (0, 0)$.

2. *On the other hand, $\mathbf{r}(t)$ is a straight line if and only if $(p(t), q(t)) = (0, 0)$*

Proof: See [34, page 218].

In normal–form, a degenerate RRMF curve is either a straight line or planar curve that satisfies (4.24) and has torsion $\tau = 0$, while a *proper* RRMF curve is a true space curve that does not satisfy (4.24) and has $\tau \neq 0$.

Chapter 5

RRMF curves of degree 5 and 7

In the present chapter we deal with some special classes of RRMF curves of degree 5 and 7. For RRMF curves of degree 5 we will study curves of types $(2,2),(2,1)$ and $(2,0)$ and for degree 7 RRMF curves we focus only on type $(3,0)$. Note that the RRMF curves of type $(2,2)$ are the "simplest" non–degenerate RRMF curves and they have been thoroughly analyzed before in [21] and [32]. However, now we analyze these curves again from another point of view. PH curves of type $(m,0)$ with $m=2,3$ satisfy (4.17) (or (4.18)) and the ERF $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ is inherently rotation–minimizing and so the rational normal–plane rotation (4.14) is not required. It has been shown that PH quintics with this property, i.e., RRMF curves of type $(2,0)$ are degenerate. Choi and Han [7] proved that the simplest non–planar curves in this category are of type $(3,0)$, i.e., PH curves of 7 degree, and characterized these curves in terms of sixteen real parameters.

The chapter is organized as follows: In Section 5.1 we review RRMF quintics of type $(2,1)$. We shall give the necessary and sufficient conditions under which a PH curve satisfying (4.15) with $\deg(a(t),b(t))=1$ and by considering the polynomial $\mathcal{A}(t)$ expressed as a product of factors. As we will see later on, the RRMF curves of type $(2,1)$ are -in some way- "connected" with a special class of PH curves of degree 7 which will be analyzed in the next sections. Also, we study RRMF curves of type $(2,0)$ and we present a necessary and sufficient condition in terms of the factorization of $\mathcal{A}(t)$ in order for a PH curve to be of type $(2,0)$. In Section 5.2 an analysis of the PH quintics curves of type $(2,2)$ by using the material of Sections 3.3–3.4 and some interesting results in terms of the roots of $\mathcal{A}(t)$ are demonstrated. Finally, in Section 5.3 we introduce PH curves of degree 7 and we focus on RRMF curves of type $(3,0)$. We present the equivalent conditions under which a PH curve is of type $(3,0)$, by considering $\mathcal{A}(t)$ in different equivalent forms. At the same time we give a parametrization of all such curves of type $(3,0)$.

5.1 RRMF curves of type $(2,1)$ and $(2,0)$

Let $\mathbf{r}(t)$ the PH curve generated by the quadratic quaternion polynomial $\mathcal{A}(t)$. The goal of this section is to present the necessary and sufficient conditions for a PH curve $\mathbf{r}(t)$ to be of type $(2,1)$ or $(2,0)$ when the polynomial $\mathcal{A}(t)$ is expressed in a factorization form

$$\mathcal{A}(t) = (t - \mathcal{C}_1)(t - \mathcal{C}_2), \tag{5.1}$$

with

$$\mathcal{C}_1 = (c_1,\, \mathbf{c}_1) = \alpha_0 + \alpha_1\,\mathbf{i} + \alpha_2\,\mathbf{j} + \alpha_3\,\mathbf{k}$$

and

$$\mathcal{C}_2 = (c_2,\, \mathbf{c}_2) = \beta_0 + \beta_1\,\mathbf{i} + \beta_2\,\mathbf{j} + \beta_3\,\mathbf{k}.$$

Let

$$w = \operatorname{scal}(\mathcal{A}(t)\,\mathbf{i}\mathcal{A}'^{*}(t)) = w_2 t^2 + w_1 t + w_0, \tag{5.2}$$

be the numerator on the left hand side in (4.15) and

$$\sigma = |\mathcal{A}(t)|^2 = t^4 + \sigma_3 t^3 + \sigma_2 t^2 + \sigma_1 t + \sigma_0 \tag{5.3}$$

be its denominator.

RRMF curves of type $(2,1)$

The quintic curve $\mathbf{r}(t)$ is of type $(2,1)$ if and only if two relatively prime linear polynomials $a(t)$, $b(t)$ exist such that

$$\frac{w}{\sigma} = \frac{a(t)b'(t) - a'(t)b(t)}{a(t)^2 + b(t)^2}$$

Since $a(t), b(t)$ are linear and relatively prime, by Lemmas 4.3 and 4.2 we may-without loss of generality-assume that $a(t) = t - a_0$ and $b(t) = b_0$ for $a_0, b_0 \in \mathbb{R}$ with $b_0 \neq 0$. Expanding (5.3) we obtain

$$\sigma_3 = -2\,(c_1 + c_2), \qquad \sigma_2 = |\mathcal{C}_1|^2 + |\mathcal{C}_2|^2 + 4\,c_1\,c_2,$$

$$\sigma_1 = -2\,(c_1|\mathcal{C}_2|^2 + c_2|\mathcal{C}_1|^2), \qquad \sigma_0 = |\mathcal{C}_1|^2|\mathcal{C}_2|^2.$$

Since

$$\mathcal{A}(t) = t^2 - (\mathcal{C}_1 + \mathcal{C}_2)t + \mathcal{C}_1\mathcal{C}_2 =$$
$$(t^2 - (c_1 + c_2)t + c_1c_2 - \mathbf{c}_1\,\mathbf{c}_2, -(\mathbf{c}_1 + \mathbf{c}_2)t + c_1\,\mathbf{c}_2 + c_2\,\mathbf{c}_1 + \mathbf{c}_1 \times \mathbf{c}_2)$$

and

$$\mathcal{A}'^*(t) = (2\,t - c_1 - c_2,\, \mathbf{c}_1 + \mathbf{c}_2)$$

by substituting in (5.2) we have that $w(t)$ has coefficients

$$\begin{aligned} w_2 &= \mathbf{i}\cdot(\mathbf{c}_1 + \mathbf{c}_2), \\ w_1 &= -2\,\mathbf{i}\cdot(c_2\,\mathbf{c}_1 + c_1\,\mathbf{c}_2 + \mathbf{c}_1 \times \mathbf{c}_2), \\ w_0 &= \mathbf{i}\cdot[|\mathcal{C}_2|^2\,\mathbf{c}_1 + |\mathcal{C}_1|^2\,\mathbf{c}_2 + 2c_2\,\mathbf{c}_1 \times \mathbf{c}_2 + 2(\mathbf{c}_1 \times \mathbf{c}_2) \times \mathbf{c}_2] \end{aligned}$$

Thus, the equality

$$\frac{w}{\sigma} = \frac{-b_0}{t^2 - 2a_0t + a_0^2 + b_0^2} \tag{5.4}$$

is equivalent to

$$\begin{aligned} b_0 &= -w_2, \\ -b_0\sigma_3 &= w_1 - 2a_0w_2, \\ -b_0\sigma_2 &= (a_0^2 + b_0^2)w_2 + w_0 - 2a_0w_1, \\ -b_0\sigma_1 &= (a_0^2 + b_0^2)w_1 - 2a_0w_0, \\ -b_0\sigma_0 &= (a_0^2 + b_0^2)w_0. \end{aligned}$$

Since $b_0 = -w_2 \neq 0$, the system becomes

$$\begin{aligned} b_0 &= -w_2, \\ a_0 &= \frac{w_1 - w_2\sigma_3}{2w_2}, \\ -b_0\sigma_2 &= (a_0^2 + b_0^2)w_2 + w_0 - 2a_0w_1, \\ -b_0\sigma_1 &= (a_0^2 + b_0^2)w_1 - 2a_0w_0, \\ -b_0\sigma_0 &= (a_0^2 + b_0^2)w_0. \end{aligned}$$

and hence we obtain that curve $\mathbf{r}(t)$ is of type $(2, 1)$ if and only if

$$a_0 = \frac{w_1 - w_2\,\sigma_3}{2\,w_2} \quad \text{and} \quad b_0 = -w_2$$

and these values must satisfy the last three equations of the system.

RRMF curves of type $(2, 0)$

The PH curve $\mathbf{r}(t)$ is of type $(2, 0)$ i.e., has a rotation-minimizing ERF if and only if $w(t) \equiv 0$. The last condition is equivalent to

$$\begin{aligned} \mathbf{i} \cdot (\mathbf{c}_1 + \mathbf{c}_2) &= 0, \\ -2\,\mathbf{i} \cdot (c_2\,\mathbf{c}_1 + c_1\,\mathbf{c}_2 + \mathbf{c}_1 \times \mathbf{c}_2) &= 0, \\ \mathbf{i} \cdot [|\mathcal{C}_2|^2\,\mathbf{c}_1 + |\mathcal{C}_1|^2\,\mathbf{c}_2 + 2c_2\,\mathbf{c}_1 \times \mathbf{c}_2 + 2(\mathbf{c}_1 \times \mathbf{c}_2) \times \mathbf{c}_2] &= 0. \end{aligned} \tag{5.5}$$

By substituting

$$\mathcal{C}_1 = \alpha_0 + \alpha_1\,\mathbf{i} + \alpha_2\,\mathbf{j} + \alpha_3\,\mathbf{k}, \quad \mathcal{C}_2 = \beta_0 + \beta_1\,\mathbf{i} + \beta_2\,\mathbf{j} + \beta_3\,\mathbf{k}$$

into (5.5) we obtain

$$\alpha_1 + \beta_1 = 0, \quad \alpha_0\beta_1 + \beta_0\alpha_1 + \alpha_2\beta_3 - \beta_2\alpha_3 = 0$$

$$(\alpha_3+\beta_3)(\alpha_0\beta_2+\beta_0\alpha_2+\alpha_3\beta_1-\beta_3\alpha_1) = (\alpha_2+\beta_2)(\alpha_0\beta_3+\beta_0\alpha_3+\alpha_1\beta_2-\beta_1\alpha_2).$$

Recall that, as it is known from [7], the only PH quintics with rotation-minimizing ERFs are planar curves.

We now express the polynomial of the form (5.1) as $\mathcal{A}(t) = u(t) + \mathbf{i}\,v(t) + \mathbf{j}\,p(t) + \mathbf{k}\,q(t)$ and we suppose that the curve $\mathbf{r}(t)$ is a straight line. By Proposition 4.1, $\mathbf{r}(t)$ is a straight line if and only if $p(t) = q(t) = 0$. In view of the above, the curve $\mathbf{r}(t)$ is a straight line of type $(2, 0)$ if and only if

$$\alpha_1 + \beta_1 = 0, \quad \alpha_0\beta_1 + \beta_0\alpha_1 + \alpha_2\beta_3 - \beta_2\alpha_3 = 0$$

$$(\alpha_3+\beta_3)(\alpha_0\beta_2+\beta_0\alpha_2+\alpha_3\beta_1-\beta_3\alpha_1) = (\alpha_2+\beta_2)(\alpha_0\beta_3+\beta_0\alpha_3+\alpha_1\beta_2-\beta_1\alpha_2)$$

$$\alpha_2 + \beta_2 = 0, \quad \alpha_0\beta_2 + \alpha_2\beta_0 + \alpha_3\beta_1 - \alpha_1\beta_3 = 0$$

$$\alpha_3 + \beta_3 = 0, \quad \alpha_0\beta_3 + \alpha_3\beta_0 + \alpha_1\beta_2 - \alpha_2\beta_1 = 0$$

The last equalities lead to

$$\frac{\alpha_0}{\beta_0} = -\frac{\alpha_1}{\beta_1} = -\frac{\alpha_2}{\beta_2} = -\frac{\alpha_3}{\beta_3},$$

i.e.,

$$\mathcal{C}_1 = \lambda\,\mathcal{C}_2^*, \quad \lambda \in \mathbb{R}.$$

Note that if $\lambda = 1$, then $\mathcal{A}(t)$ is non-primitive polynomial which is not the case.

The above discussion is summarized as follows.

Proposition 5.1. *Let* $\mathcal{A}(t) = (t-\mathcal{C}_1)(t-\mathcal{C}_2)$ *with* $\mathcal{C}_1 = \alpha_0+\alpha_1\,\mathbf{i}+\alpha_2\,\mathbf{j}+\alpha_3\,\mathbf{k}$, $\mathcal{C}_2 = \beta_0+\beta_1\,\mathbf{i}+\beta_2\,\mathbf{j}+\beta_3\,\mathbf{k}$. *Set* $w(t) = \mathrm{scal}(\mathcal{A}(t)\,\mathbf{i}\mathcal{A}'^*(t)) = w_2t^2+w_1t+w_0$ *and* $\sigma(t) = |\mathcal{A}(t)|^2 = t^4+\sigma_3t^3+\sigma_2t^2+\sigma_1t+\sigma_0$. *Then, the PH curve* $\mathbf{r}(t)$ *generated by the polynomial* $\mathcal{A}(t)$ *is*

1. *of type* $(2,1)$ *if and only if the system*

$$\begin{aligned}
-b_0\,\sigma_2 &= (a_0^2+b_0^2)\,w_2 + w_0 - 2\,a_0w_1,\\
-b_0\,\sigma_1 &= (a_0^2+b_0^2)\,w_1 - 2\,a_0w_0,\\
-b_0\,\sigma_0 &= (a_0^2+b_0^2)\,w_0
\end{aligned}$$

has the solution

$$(a_0,b_0) = (\frac{w_1-w_2\sigma_3}{2w_2}, -w_2).$$

2. *of type* $(2,0)$ *i.e., has a rotation-minimizing ERF if and only if the following equalities hold*

$$\alpha_1+\beta_1 = 0, \quad \alpha_0\beta_1+\beta_0\alpha_1+\alpha_2\beta_3-\beta_2\alpha_3 = 0$$

$$(\alpha_3+\beta_3)(\alpha_0\beta_2+\beta_0\alpha_2+\alpha_3\beta_1-\beta_3\alpha_1) = (\alpha_2+\beta_2)(\alpha_0\beta_3+\beta_0\alpha_3+\alpha_1\beta_2-\beta_1\alpha_2).$$

Moreover, this curve is a straight line if and only if

$$\mathcal{C}_1 = \lambda\,\mathcal{C}_2^*, \quad \lambda\in\mathbb{R},\ \lambda\neq 1.$$

Example 5.1.1 Let

$$\mathcal{A}(t) \;=\; (t+\frac{2}{9}\mathbf{i}+\frac{14}{9}\mathbf{j}+\frac{5}{9}\mathbf{k})\,(t-2+\frac{7}{9}\mathbf{i}+\frac{4}{9}\mathbf{j}+\frac{4}{9}\mathbf{k})$$

be a quaternion polynomial which defines a PH quintic curve. We can easily see that

$$w_0 = -\frac{25}{9}, \quad w_1 = 0, \quad w_2 = -1,$$

$$\sigma_0 = \frac{125}{9}, \quad \sigma_1 = -\frac{100}{9} \quad \sigma_2 = \frac{70}{9}, \quad \sigma_3 = -4$$

$$a_0 = 2, \quad b_0 = 1$$

and the system of Proposition 5.1 is verified by the values of a_0, b_0. Thus, $\mathcal{A}(t)$ defines an RRMF curve of type $(2,1)$.

Example 5.1.2 Let

$$\mathcal{A}(t) = (t - \frac{1}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j} - \frac{1}{2}\mathbf{k})^2.$$

Then, applying Proposition 5.1 we take $a_0 = 0$ and $b_0 = -2$ and the system is verified. Thus, $\mathcal{A}(t)$ defines an RRMF curve of type $(2, 1)$.

5.2 Analysis of quintic RRMF curves of type $(2, 2)$

In this Section we apply the results of Sections 3.3 and 3.4 to analyze certain root properties of the quadratic quaternion polynomials that generate quintic RRMF curves. As it was shown in [21], the satisfaction of the constraint (4.21) by the coefficient of the quadratic $\mathcal{A}(t)$, expressed in Bernstein–form (4.19), is sufficient and necessary for the PH quintic to be an RRMF curve of type $(2, 2)$. In view of (4.21) one can ask whether the RRMF quintics can be alternatively characterized by means of the special root structure of the quaternion polynomials that generate them. The method which is used in [21] to derive (4.21) does not easily extend to degree 7 or higher–order degree PH curves, and perhaps a root–structure characterization of the RRMF curves may offer an alternative approach to the study of the higher–order curves and thus the motivation of our study.

As we shall see, the algorithm presented in Section 3.4 is used here to characterize these polynomials in terms of their root structure. Moreover, we shall prove that polynomials with a double quaternion root generate degenerate quintic curves and for polynomials with distinct roots we shall give a closed–form description of the roots in terms of uniform scale factor, a quaternion with unit vector part, and a parameter $\tau \in [-1, +1]$.

Consider the PH curve defined by the quadratic quaternion polynomial $\mathcal{A}(t)$ expressed in (4.19). To apply the method of Sections 3.3 and 3.4 to determine the roots of (4.19) we write it in power form

$$\mathcal{A}(t) = \mathcal{D}\,t^2 + \mathcal{B}\,t + \mathcal{C} \tag{5.6}$$

where

$$\mathcal{A}_0 = \mathcal{C}\,, \qquad \mathcal{A}_1 = \tfrac{1}{2}\mathcal{B} + \mathcal{C}\,, \qquad \mathcal{A}_2 = \mathcal{D} + \mathcal{B} + \mathcal{C}\,. \tag{5.7}$$

We first give the next Lemma.

Lemma 5.1. *If the polynomial $\mathcal{A}(t)$ is represented in power form, the condition*

$$\mathrm{vect}(\mathcal{D}\,\mathbf{i}\,\mathcal{C}^*) \;=\; \tfrac{1}{4}\,\mathcal{B}\,\mathbf{i}\,\mathcal{B}^* \tag{5.8}$$

on its coefficients is sufficient and necessary for the PH quintic specified by (4.5) and (5.6) to be an RRMF curve of type $(2,2)$.

Proof: We express $\mathcal{A}(t)$ in the Bernstein form (4.19). By Theorem 4.2 a sufficient and necessary condition for the PH quintic to be an RRMF curve satisfying (4.15) with $\deg(a(t),b(t)) = 2$ is (4.21). Substituting (5.7) into (4.21) and simplifying we obtain the result. ∎

By Lemma 4.3 we can assume, without loss of generality, that $\mathcal{D} = (1,\mathbf{0})$ and thus $\mathcal{A}(t)$ is expressed in normal form i.e., is monic. This assumption does not influence the RRMF nature of a given PH curve, and does not change the roots of (5.6) [22]. In the following we deal with curves specified by

$$\mathcal{A}(t) = t^2 + \mathcal{B}\,t + \mathcal{C} \tag{5.9}$$

and (4.5). We write the scalar and vector parts of $\mathcal{B}$ as b and $\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$, and of $\mathcal{C}$ as c and $\mathbf{c} = c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k}$.

Recall that here we shall focus only on the case $\deg(a(t),b(t)) = 2$ or type $(2,2)$ curves.

Proposition 5.2. *A PH quintic defined by (4.5) and (5.9) is an RRMF curve of type* $(2,2)$ *if and only if* $\mathcal{C} = (c,\mathbf{c})$ *can be expressed as*

$$\mathcal{C} \;=\; \left(\,\tfrac{1}{4}(b^2 - |\mathbf{b}|^2 + 2\,b_x^2)\,,\,\tfrac{1}{2}(\xi\,\mathbf{i} + b\,\mathbf{b} + b_x\,\mathbf{i}\times\mathbf{b})\,\right)\,. \tag{5.10}$$

where $\mathcal{B} = (b,\mathbf{b})$ *and* ξ *is a real parameter.*

Proof: Using (5.8), the PH defined by (5.9) is an RRMF curve of type $(2,2)$ if and only if

$$c\,\mathbf{i} + \mathbf{c}\times\mathbf{i} \;=\; \tfrac{1}{4}\left(\,(b^2 - |\mathbf{b}|^2)\,\mathbf{i} + 2\,b_x\,\mathbf{b} + 2\,b\,\mathbf{b}\times\mathbf{i}\,\right). \tag{5.11}$$

Taking the dot product of both sides with $\mathbf{i}$, we get

$$c \;=\; \tfrac{1}{4}\,(b^2 - |\mathbf{b}|^2 + 2\,b_x^2)\,, \tag{5.12}$$

and substituting this into (5.11) we obtain

$$(\mathbf{c} - \tfrac{1}{2}\, b\, \mathbf{b} - \tfrac{1}{2}\, b_x\, \mathbf{i} \times \mathbf{b}) \times \mathbf{i} \,=\, \mathbf{0}\,.$$

Hence, we get

$$\mathbf{c} \,=\, \tfrac{1}{2}\,(\xi\,\mathbf{i} + b\,\mathbf{b} + b_x\,\mathbf{i} \times \mathbf{b})\,, \tag{5.13}$$

where ξ is a real parameter and the quaternion coefficient $\mathcal{C} = (c, \mathbf{c})$ has the stated form (5.10). ■

Now any linear or planar locus is trivially an RRMF curve, and since we are only interested in space curves, we first identify instances of $\mathbf{b}$ and ξ that define straight lines or plane curves. These cases are discarded in analyzing the roots of the polynomials $\mathcal{A}(t)$ that generate spatial RRMF curves.

Proposition 5.3. *With $\mathcal{C}$ given by* (5.10)*, substituting (5.9) into* (4.5) *generates straight lines if and only if $(b_y, b_z) = (0,0)$, and planar curves other than straight lines if and only if $b_x = \xi = 0$ and $(b_y, b_z) \neq (0,0)$.*

Proof: With $\mathcal{C}$ specified by (5.10), the components of $\mathcal{A}(t) = u(t) + \mathbf{i}\,v(t) + \mathbf{j}\,p(t) + \mathbf{k}\,q(t)$ are given by

$$\begin{aligned}
u(t) \,&=\, t^2 \,+\, b\,t \,+\, \tfrac{1}{4}\,(b^2 + b_x^2 - b_y^2 - b_z^2)\,, \\
v(t) \,&=\, b_x t \,+\, \tfrac{1}{2}\,(\xi + b\,b_x)\,, \\
p(t) \,&=\, b_y t \,+\, \tfrac{1}{2}\,(b\,b_y - b_x b_z)\,, \\
q(t) \,&=\, b_z t \,+\, \tfrac{1}{2}\,(b\,b_z + b_x b_y)\,.
\end{aligned}$$

Since $\deg(u) = 2$ and $\deg(v, p, q) < 2$, by Lemma 4.3 we have that $\mathcal{A}(t)$ is in normal form. Then, by Proposition 4.1, the curve defined by (4.5) degenerates to a straight line if and only if $(p(t), q(t)) = (0,0)$, for each t and to a planar curve other than a straight line if and only if (4.24) holds with $(p(t), q(t)) \neq (0,0)$, for each t. Obviously, $(p(t), q(t)) \equiv (0,0)$ if and only if $(b_y, b_z) = (0,0)$ and thus this is a sufficient and necessary condition for the curve to be a straight line. From the last equivalence we have that $(p(t), q(t)) \not\equiv (0,0)$ if $(b_y, b_z) \neq (0,0)$, and using `MAPLE` we find that (4.24) becomes

$$-\,\frac{b_y^2 + b_z^2}{32}\,(c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t + c_0) \,\equiv\, 0\,, \tag{5.14}$$

where

$$\begin{aligned} c_4 &= 48\, b_x \,, \quad c_3 \,=\, 32\,(\xi + 3\, b\, b_x)\,, \\ c_2 &= 24\,(2\, b\, \xi + 3\, b^2 b_x + b_x^3)\,, \quad c_1 \,=\, 24\,(b^2 + b_x^2)(\xi + b\, b_x)\,, \\ c_0 &= \;4\, b_x \xi^2 \,+\, 4\, b\,(b^2 + 3\, b_x^2)\, \xi \,+\, b_x\,[\, 3\, b^4 + 6\, b^2 b_x^2 - b_x^4 + (b_y^2 + b_z^2)^2\,]\,. \end{aligned}$$

If (5.14) is satisfied, then from c_4 we must have $b_x = 0$. Substituting $b_x = 0$, we find using `MAPLE` that this condition reduces to

$$-\,\frac{b_y^2 + b_z^2}{8}\,\xi\,(2t + b)^3 \;\equiv\; 0\,.$$

Since $b_y^2 + b_z^2 \neq 0$, the condition for a planar curve (other than a straight line) corresponds to $b_x = \xi = 0$. Clearly, for $b_x = \xi = 0$ and $(b_y, b_z) \neq (0, 0)$ condition (4.24) is satisfied. ∎

We have the following remarks concerning the degeneration of an RRMF of type $(2, 2)$ to a straight line, or a plane curve other than a straight line.

Remark 5.1. The condition $b_y = b_z = 0$ for degeneration to a straight line is automatically satisfied if $\mathbf{b} = \mathbf{0}$.

Remark 5.2. When $\mathcal{C}$ is given by (5.10) with $b_y = b_z = 0$, the polynomial (5.9) reduces to

$$\mathcal{A}(t) \;=\; \big(\, t^2 + b\,t + \tfrac{1}{4}(b^2 + b_x^2)\,,\, [\, b_x t + \tfrac{1}{2}(\xi + b\, b_x)\,]\,\mathbf{i}\,\big)\,.$$

Thus, degeneration to a straight line occurs when $\mathcal{A}(t)$ is a *complex* polynomial. If $b_x = \xi = 0$, on the other hand, we have

$$\mathcal{A}(t) \;=\; \big(\, t^2 + b\,t + \tfrac{1}{4}(b^2 - b_y^2 - b_z^2)\,,\, (t + \tfrac{1}{2}\, b)(b_y\mathbf{j} + b_z\mathbf{k})\,\big)\;,$$

so degeneration to a plane curve other than a straight line occurs when the vector part of $\mathcal{A}(t)$ has no $\mathbf{i}$ component.

From Proposition 5.2, it is evident that the roots of a monic quadratic quaternion polynomial $\mathcal{A}(t)$ that defines an RRMF curve of type $(2, 2)$ depend only on the quaternion coefficient $\mathcal{B} = (b, \mathbf{b})$ and the real parameter

ξ. The coefficients (3.7) of the cubic (3.8) can be expressed in terms of b, $\mathbf{b}$, and ξ as

$$a_2 = |\mathbf{b}|^2 + 2\,b_x^2\,, \qquad a_1 = b_x^2(|\mathbf{b}|^2 + b_x^2) - \xi^2\,, \qquad a_0 = -\,b_x^2\,\xi^2\,, \quad (5.15)$$

and we observe that these coefficients do *not* depend on b. In fact, the cubic (3.8) has a very special structure when the quaternion polynomial (5.9) satisfies condition (5.8), and thus generates a quintic RRMF curve of type $(2,2)$ through (4.5).

Lemma 5.2. *For a monic quadratic quaternion polynomial (5.9) with $\mathcal{C}$ given by* (5.10)*, the cubic equation* (3.8) *specified by the coefficients* (5.15) *admits the factorization*

$$(x^2 + (|\mathbf{b}|^2 + b_x^2)x - \xi^2)\,(x + b_x^2) \,=\, 0\,. \tag{5.16}$$

Proof: Expanding (5.16) yields the cubic (3.8) with the coefficients (5.15). ∎

Thus, the computation of the roots of the quadratic quaternion polynomials that generate RRMF curves of type $(2,2)$ does not require solution of a cubic equation, indicating a special structure to these quaternion roots. If $b_x \neq 0$ and $\xi \neq 0$, the only positive root of (5.16) is

$$\rho \,=\, \sqrt{\xi^2 + \tfrac{1}{4}(|\mathbf{b}|^2 + b_x^2)^2} \,-\, \tfrac{1}{2}(|\mathbf{b}|^2 + b_x^2) \tag{5.17}$$

and this determines two generic quaternion roots of $\mathcal{A}(t)$, specified by

$$\mathcal{Q} \,=\, \left(-\,\frac{b}{2}\,,\,\frac{\mathbf{b}\times\mathbf{c}}{\rho + |\mathbf{b}|^2} - \frac{\mathbf{b}}{2}\right) \,\pm\, \frac{1}{\sqrt{\rho}}\left(\frac{\rho}{2}\,,\,\frac{b\,\mathbf{b}}{2} - \frac{\rho\,\mathbf{c} + (\mathbf{b}\cdot\mathbf{c})\,\mathbf{b}}{\rho + |\mathbf{b}|^2}\right) \tag{5.18}$$

where ρ is the unique positive root of (3.8) with $a_0 \neq 0$.

If $b_x = 0$ or $\xi = 0$, however, then $a_0 = 0$ and $\mathcal{A}(t)$ may possess singular roots. We first consider the case where $x = 0$ is a root of (5.16). The following result characterizes the instance $b_x = \xi = 0$.

Lemma 5.3. *If the polynomial (5.9) with $\mathcal{C}$ given by* (5.10) *has a double root, the RRMF curve of type* $(2,2)$ *defined by* (4.5) *degenerates to a planar curve or a straight line.*

Proof: With $\mathcal{C}$ given by (5.10), the conditions (3.18) for a double root become

$$b_x\xi \,=\, 0 \qquad \text{and} \qquad \xi^2 \,+\, 2\,bb_x\xi \,-\, b_x^2(2b_x^2 + b_y^2 + b_z^2) \,=\, 0\,.$$

The first condition implies that $b_x = 0$ or $\xi = 0$, whereas the second condition cannot be satisfied if $b_x = 0 \neq \xi$ or $b_x \neq 0 = \xi$, so we must have $b_x = \xi = 0$. By Proposition 5.3, the curve defined by (4.5) is a straight line when $(b_y, b_z) = (0, 0)$ and a planar curve other than a straight line when $(b_y, b_z) \neq (0, 0)$ — the double root of $\mathcal{A}(t)$ in the former case is $\mathcal{Q} = (-\frac{1}{2}\,b, \mathbf{0})$, and in the latter is $\mathcal{Q} = (-\,\frac{1}{2}\,b, -\frac{1}{2}\,(b_y\,\mathbf{j} + b_z\,\mathbf{k}))$. ∎

Consider now the cases in which just one of b_x or ξ is zero. If $b_x = 0 \neq \xi$, we obtain $c = \frac{1}{4}\,(b^2 - |\mathbf{b}|^2)$ and $\mathbf{c} = \frac{1}{2}\,(\xi\,\mathbf{i} + b\,\mathbf{b})$ from (5.12) and (5.13), and thus $|\mathbf{b}|^4 + 4\,c\,|\mathbf{b}|^2 - 4\,|\mathbf{c}|^2 = -\,\xi^2$. Since condition (3.16) is obviously not satisfied, there are no singular roots. On the other hand, the quadratic factor in (5.16) has a single positive real root

$$\rho \,=\, \sqrt{\xi^2 + \tfrac{1}{4}\,|\mathbf{b}|^4} - \tfrac{1}{2}\,|\mathbf{b}|^2\,,$$

corresponding to the specialization $b_x = 0$ of (5.17). Then, there are two generic quaternion roots, defined with this ρ value in (3.11). Finally, when $\xi = 0 \neq b_x$, equation (3.14) reduces to

$$|\mathbf{b}|^2\left[\,|\mathbf{b}|^4(\gamma + \tfrac{1}{2})^2 \,-\, \tfrac{1}{4}\,b_x^2(|\mathbf{b}|^2 + b_x^2)\,\right] \,=\, 0\,,$$

and assuming that $\mathbf{b} \neq \mathbf{0}$ (See Remark 5.1) it has the two solutions

$$\gamma \,=\, \frac{-\,|\mathbf{b}|^2 \pm b_x\sqrt{|\mathbf{b}|^2 + b_x^2}}{2\,|\mathbf{b}|^2}\,.$$

Then, expression (3.17) for the two singular roots reduces to

$$\mathcal{Q} \,=\, \left(-\frac{b}{2}\,,\, \frac{b_x|\mathbf{b}|^2\mathbf{i} - (|\mathbf{b}|^2 + b_x^2)\,\mathbf{b}}{2\,|\mathbf{b}|^2}\right) \pm \left(0\,,\, \frac{b_x\sqrt{|\mathbf{b}|^2 + b_x^2}\,\mathbf{b}}{2\,|\mathbf{b}|^2}\right)\,. \tag{5.19}$$

For the generic roots, with the positive root of (5.16) defined by (5.17) when $(b_x, \xi) \neq (0, 0)$, we have

$$\mathbf{b}\cdot\mathbf{c} \,=\, \tfrac{1}{2}\,(b_x\xi + b\,|\mathbf{b}|^2)\,,$$

$$\mathbf{b}\times\mathbf{c} \;=\; \tfrac{1}{2}\left(\xi\,\mathbf{b}\times\mathbf{i}+b_x\mathbf{b}\times(\mathbf{i}\times\mathbf{b})\right) \;=\; \tfrac{1}{2}\left(b_x|\mathbf{b}|^2\mathbf{i}-b_x^2\mathbf{b}-\xi\,\mathbf{i}\times\mathbf{b}\right).$$

Substituting into (3.11) and simplifying gives the roots as

$$\begin{aligned}\mathcal{Q} \;&=\; \left(-\frac{b}{2}\,,\,\frac{b_x|\mathbf{b}|^2\,\mathbf{i}-b_x^2\,\mathbf{b}-\xi\,\mathbf{i}\times\mathbf{b}}{2(\rho+|\mathbf{b}|^2)}-\frac{\mathbf{b}}{2}\right)\\ &\pm\;\frac{1}{\sqrt{\rho}}\left(\frac{\rho}{2}\,,\,-\frac{\rho\,\xi\,\mathbf{i}+b_x\xi\,\mathbf{b}+\rho\,b_x\,\mathbf{i}\times\mathbf{b}}{2(\rho+|\mathbf{b}|^2)}\right).\end{aligned} \tag{5.20}$$

Lemma 5.4. *The singular roots* (5.19) *are the formal limit of the generic roots* (5.20)*, as* $\xi\to 0$*.*

Proof: First, note from (5.17) that $\rho\to 0$ as $\xi\to 0$, and otherwise ρ increases monotonically with $|\xi|$. Setting $\xi=\rho=0$ in the first term of (5.20), it clearly reduces to the first term of (5.19). Likewise, the scalar part of the second term of (5.20) is zero when $\rho=0$, and thus agrees with the scalar part of the second term of (5.19). The vector part of the second term in (5.20) requires more analysis. First, it is clear that the $\mathbf{i}$ and $\mathbf{i}\times\mathbf{b}$ terms in this vector part vanish as $\rho\to 0$. For the $\mathbf{b}$ term, we use (5.17) to write ξ in terms of ρ as

$$\xi \;=\; \pm\sqrt{\rho^2+(|\mathbf{b}|^2+b_x^2)\rho}\,,$$

and we have

$$\pm\lim_{\rho\to 0}\frac{b_x\xi\,\mathbf{b}}{2\sqrt{\rho}\,(\rho+|\mathbf{b}|^2)} \;=\; \pm\lim_{\rho\to 0}\frac{b_x\sqrt{\rho+|\mathbf{b}|^2+b_x^2}\,\mathbf{b}}{2(\rho+|\mathbf{b}|^2)} \;=\; \pm\,\frac{b_x\sqrt{|\mathbf{b}|^2+b_x^2}\,\mathbf{b}}{2\,|\mathbf{b}|^2}\,.$$

Hence, the generic roots (5.20) converge to the singular roots (5.19) as $\rho\to 0$ (and hence $\xi\to 0$). ■

Lemma 5.5. *For each* ξ *value, the roots* (5.20) *scale linearly with the quaternion coefficient* $\mathcal{B}=(b,\mathbf{b})$*.*

Proof: We invoke the parameter transformation $\xi\to\psi$ defined by

$$\xi \;=\; \tfrac{1}{2}\,(|\mathbf{b}|^2+b_x^2)\,\tan\psi\,, \tag{5.21}$$

specifying a one–to–one map between $\xi\in(-\infty,+\infty)$ and $\psi\in(-\frac{1}{2}\pi,+\frac{1}{2}\pi)$. Then from (5.17) we have

$$\rho \;=\; \tfrac{1}{2}\,(|\mathbf{b}|^2+b_x^2)\,(\sec\psi-1)\,. \tag{5.22}$$

Hence $\xi \to \lambda^2 \xi$ and $\rho \to \lambda^2 \rho$ for each ψ when $\mathcal{B} = (b, \mathbf{b}) \to \lambda \mathcal{B} = (\lambda b, \lambda \mathbf{b})$ and we see that the roots (5.20) then scale as $\mathcal{Q} \to \lambda \mathcal{Q}$. ∎

By Lemma 5.5, a particular scaling can be imposed on $\mathcal{B} = (b, \mathbf{b})$ without altering the roots of equation (3.1) in an essential manner. For simplicity, we assume henceforth that[1] $|\mathbf{b}| = 1$, i.e.,

$$b_x^2 + b_y^2 + b_z^2 \,=\, 1\,. \tag{5.23}$$

Now setting $\tau = \tan \frac{1}{2}\psi \in [\,-1, +1\,]$ we have

$$\tan \psi \,=\, \frac{2\,\tau}{1 - \tau^2} \qquad \text{and} \qquad \sec \psi \,=\, \frac{1 + \tau^2}{1 - \tau^2}\,, \tag{5.24}$$

and we note that $\rho = \xi\,\tau$. Using (5.21)–(5.24), the scalar part of (5.20) can then be written as

$$q \,=\, \frac{1}{2}\left(-\,b \pm |\tau| \sqrt{\frac{1 + b_x^2}{1 - \tau^2}}\,\right), \tag{5.25}$$

while the vector part reduces to

$$\begin{aligned} \mathbf{q} \,=\,& \frac{-\,b_x(b_x^2 + \tau^2)\,\mathbf{i} - (1 + b_x^2)\,[\,(b_y - b_z\tau)\,\mathbf{j} + (b_z + b_y\tau)\,\mathbf{k}\,]}{2\,(1 + b_x^2\tau^2)} \\ & \mp\, \mathrm{sign}(\tau)\sqrt{\frac{1 + b_x^2}{1 - \tau^2}}\,\frac{(b_x^2 + \tau^2)\,\mathbf{i} + b_x(1 - \tau^2)\,[\,(b_y - b_z\tau)\,\mathbf{j} + (b_z + b_y\tau)\,\mathbf{k}\,]}{2\,(1 + b_x^2\tau^2)}\,. \end{aligned} \tag{5.26}$$

Note that $\mathbf{q}$ does not depend on b. When $\tau = 0$, expressions (5.25)–(5.26) agree with the singular roots (5.19) under the assumption $|\mathbf{b}| = 1$. As $\tau \to \pm 1$, on the other hand, $q \to \pm\infty$ and $\mathbf{q}$ increases without bound in the direction $\pm\mathbf{i}$. The preceding results may be summarized as follows.

Proposition 5.4. *The quadratic quaternion polynomials that generate quintic RRMF curves of type* $(2, 2)$ *are characterized by roots of the form* $\mathcal{Q} = \lambda(q, \mathbf{q})$ *where* $\lambda > 0$ *is a scale factor, while* q *and* $\mathbf{q}$ *depend on a real value* b*, a unit vector* $\mathbf{b}$*, and a real parameter* $\tau \in [\,-1, +1\,]$ *through expressions* (5.25)*–*(5.26).

The above arguments are illustrated by means of the following example.

[1] Recall from Remark 5.1 that we require $\mathbf{b} \neq \mathbf{0}$ for a true space curve.

Example 5.2.1 With the choices

$$b = -1\,, \qquad \mathbf{b} = \frac{\mathbf{j}+\mathbf{k}}{\sqrt{2}}\,, \qquad \xi = 1\,,$$

Proposition 5.2 gives, for an RRMF curve of type $(2,2)$,

$$c = 0 \qquad \text{and} \qquad \mathbf{c} = \frac{\sqrt{2}\,\mathbf{i}-\mathbf{j}-\mathbf{k}}{2\sqrt{2}}\,.$$

With $\mathcal{B} = (b,\mathbf{b})$, $\mathcal{C} = (c,\mathbf{c})$, the polynomial $\mathcal{A}(t) = u(t)+v(t)\,\mathbf{i}+p(t)\,\mathbf{j}+q(t)\,\mathbf{k}$ defined by (5.9) has the components

$$u(t) = t^2 - t\,, \quad v(t) = \frac{1}{2}\,, \quad p(t) = \frac{2\,t-1}{2\sqrt{2}}\,, \quad q(t) = \frac{2\,t-1}{2\sqrt{2}}\,,$$

and generates the Pythagorean hodograph

$$x'(t) = t^4 - 2\,t^3 + t\,, \quad y'(t) = \frac{4\,t^3 - 6\,t^2 + 4\,t - 1}{2\sqrt{2}}\,, \quad z'(t) = \frac{-\,4\,t^3 + 6\,t^2 - 1}{2\sqrt{2}}$$

which satisfies (4.3)

with $\sigma(t) = t^4 - 2\,t^3 + 2\,t^2 - t + \frac{1}{2}$, and (4.15) with $a(t) = t^2 - t + \frac{1}{2}$, $b(t) = \frac{1}{2}$. Since $(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}''' \not\equiv 0$, the resulting RRMF quintic is a true space curve. From (5.21) and (5.24) we obtain

$$\tan\psi = 2 \qquad \text{and} \qquad \tau = \frac{\sqrt{5}-1}{2}\,,$$

so the scalar and vector parts of the roots $\mathcal{Q} = (q,\mathbf{q})$ become

$$q = \frac{\sqrt{2} \pm \sqrt{\sqrt{5}-1}}{2\sqrt{2}}\,, \quad \mathbf{q} = \frac{\mp(\sqrt{5}-1)^{3/2}\,\mathbf{i} - (3-\sqrt{5})\,\mathbf{j} - (1+\sqrt{5})\,\mathbf{k}}{4\sqrt{2}}\,,$$

and one can verify that these roots satisfy (3.1) with

$$\mathcal{B} = \left(-1, \frac{\mathbf{j}+\mathbf{k}}{\sqrt{2}}\right) \qquad \text{and} \qquad \mathcal{C} = \left(0, \frac{\sqrt{2}\,\mathbf{i}-\mathbf{j}-\mathbf{k}}{2\sqrt{2}}\right)\,.$$

5.3 RRMF curves of type $(3,0)$

In this section we consider the circumstances under which spatial PH curves possess rotation–minimizing ERFs i.e., $\alpha(t)$, $\beta(t)$ are such that the RRMF condition (4.16) is satisfied with $\mathrm{Im}(\overline{\boldsymbol{\alpha}}\boldsymbol{\alpha}' + \overline{\boldsymbol{\beta}}\boldsymbol{\beta}') \equiv 0$, and hence $\mathbf{w}(t)$ is just a constant. As it is mentioned, for such curves the ERF is itself rotation-minimizing, and hence the normal-plane rotation (4.14) is not required. As originally shown by Choi and Han [7], the simplest non-planar curves in this category occur for $m = 3$, and characterized these RRMF curves of type $(3, 0)$ in terms of sixteen real parameters. A more concise characterization in terms of the Hopf map form is derived here, and a characterization in terms of the coefficients of the real polynomials $u(t), v(t), p(t), q(t)$ definining their hodograph is presented as well. An algorithm to construct rotation–minimizing rigid body motion interpolants using RRMF curves of type $(3, 0)$ is also given in [26]. Furthermore, we shall give a much simpler characterization of these degree 7 curves by a reduction to canonical form. Note that the reduction to canonical form corresponds to a scaling/rotation of the hodograph and thus it does not affect satisfaction of (4.15) or (4.16) [21, 34].

5.3.1 Necessary and sufficient conditions in terms of Hopf map representation

Our interest here is to find necessary and sufficient conditions for a PH curve of type $(3, 0)$ when the quaternion polynomial $\mathcal{A}(t)$ is expressed in the Hopf form

$$\mathcal{A}(t) = \boldsymbol{\alpha}(t) + \mathbf{k}\boldsymbol{\beta}(t) \tag{5.27}$$

with

$$\boldsymbol{\alpha}(t) = \boldsymbol{\alpha}_0 b_0^3(t) + \boldsymbol{\alpha}_1 b_1^3(t) + \boldsymbol{\alpha}_2 b_2^3(t) + \boldsymbol{\alpha}_3 b_3^3(t)$$

$$\boldsymbol{\beta}(t) = \boldsymbol{\beta}_0 b_0^3(t) + \boldsymbol{\beta}_1 b_1^3(t) + \boldsymbol{\beta}_2 b_2^3(t) + \boldsymbol{\beta}_3 b_3^3(t) \tag{5.28}$$

This curve is to have rotation - minimizing frame ERF if and only if

$$\mathrm{Im}(\overline{\boldsymbol{\alpha}}\boldsymbol{\alpha}' + \overline{\boldsymbol{\beta}}\boldsymbol{\beta}') \equiv 0.$$

We substitute the above cubic complex polynomials into the last equation and by using the rules of multiplication and derivative [29] we get

$$\begin{aligned}\mathrm{Im}(\overline{\boldsymbol{\alpha}}\boldsymbol{\alpha}'+\overline{\boldsymbol{\beta}}\boldsymbol{\beta}') &= 3\mathrm{Im}(\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1+\overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1)b_0^4(t)+\frac{3}{2}\mathrm{Im}(\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_2+\overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_2)b_1^4(t)\\ &+\frac{1}{2}[3\mathrm{Im}(\overline{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_2+\overline{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2)+\mathrm{Im}(\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_3+\overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_3)]b_2^4(t)\\ &+\frac{3}{2}\mathrm{Im}(\overline{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_3+\overline{\boldsymbol{\beta}}_1\boldsymbol{\beta}_3)b_3^4(t)+3\mathrm{Im}(\overline{\boldsymbol{\alpha}}_2\boldsymbol{\alpha}_3+\overline{\boldsymbol{\beta}}_2\boldsymbol{\beta}_3)b_4^4(t).\end{aligned}$$

and hence

$$\begin{aligned}\mathrm{Im}(\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1+\overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1) = \mathrm{Im}(\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_2+\overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_2) = 0\\ 3\mathrm{Im}(\overline{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_2+\overline{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2)+\mathrm{Im}(\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_3+\overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_3) = 0\\ \mathrm{Im}(\overline{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_3+\overline{\boldsymbol{\beta}}_1\boldsymbol{\beta}_3) = \mathrm{Im}(\overline{\boldsymbol{\alpha}}_2\boldsymbol{\alpha}_3+\overline{\boldsymbol{\beta}}_2\boldsymbol{\beta}_3) = 0\end{aligned} \tag{5.29}$$

These equations impose five reals constraints on the sixteen degrees of freedom in the complex coefficients $(\boldsymbol{\alpha}_i, \boldsymbol{\beta}_i)$ for $i = 0, \ldots, 3$.

Remark 5.3. For the quaternion form (4.5), we can also use a cubic polynomial

$$\mathcal{A}(t) = \mathcal{A}_0 b_0^3(t) + \mathcal{A}_1 b_1^3(t) + \mathcal{A}_2 b_2^3(t) + \mathcal{A}_3 b_3^3(t).$$

In terms of its quaternions coefficients, the conditions (5.29) for a rotation-minimizing ERF become

$$\begin{aligned}\mathrm{scal}(\mathcal{A}_0\mathbf{i}\mathcal{A}_1^*) = \mathrm{scal}(\mathcal{A}_0\mathbf{i}\mathcal{A}_2^*) = 0,\\ 3\mathrm{scal}(\mathcal{A}_1\mathbf{i}\mathcal{A}_2^*)+\mathrm{scal}(\mathcal{A}_0\mathbf{i}\mathcal{A}_3^*) = 0,\\ \mathrm{scal}(\mathcal{A}_1\mathbf{i}\mathcal{A}_3^*) = \mathrm{scal}(\mathcal{A}_2\mathbf{i}\mathcal{A}_3^*) = 0.\end{aligned} \tag{5.30}$$

The conditions (5.29) or (5.30) that identify rational ERFs are equivalent to the constraints defined by equations (32)-(33) in [7].

We now derive an alternative to the characterization (5.29), in terms of one real and two complex constraints on the coefficients of (5.28). For brevity, we assume that $\mathrm{Im}(\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_3) \neq 0$. This assumption is justified in Remark 5.5.

Proposition 5.5. *If $Im(\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_3) \neq 0$, the conditions (5.29) identifying rotation minimizing ERFs on degree 7 PH curves are equivalent to*

$$\boldsymbol{\alpha}_1 = \frac{Im(\overline{\boldsymbol{\beta}}_3\boldsymbol{\beta}_1)\boldsymbol{\alpha}_0 - Im(\overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1)\boldsymbol{\alpha}_3}{Im(\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_3)} \tag{5.31}$$

$$\boldsymbol{\alpha}_2 = \frac{Im(\overline{\boldsymbol{\beta}}_3\boldsymbol{\beta}_2)\boldsymbol{\alpha}_0 - Im(\overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_2)\boldsymbol{\alpha}_3}{Im(\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_3)} \tag{5.32}$$

$$Im(\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_3 + \overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_3)Im(\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_3 + 3\overline{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2) = 0 \tag{5.33}$$

Proof: We consider that conditions (5.29) are satisfied with $\mathrm{Im}(\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_3) \neq 0$. Then by setting $\boldsymbol{\alpha}_i = u_i + v_i\,\mathbf{i}$ and $\boldsymbol{\beta}_i = q_i + p_i\,\mathbf{i}$ for $i = 0, \ldots, 3$ into the first and fourth conditions in (5.29) we obtain

$$u_1 = \frac{(q_3p_1 - q_1p_3)u_0 - (q_0p_1 - q_1p_0)u_3}{u_0v_3 - u_3v_0},$$

$$v_1 = \frac{(q_3p_1 - q_1p_3)v_0 - (q_0p_1 - q_1p_0)v_3}{u_0v_3 - u_3v_0},$$

and since $q_3p_1 - q_1p_3 = \mathrm{Im}(\overline{\boldsymbol{\beta}}_3\boldsymbol{\beta}_1)$, $q_0p_1 - q_1p_0 = \mathrm{Im}(\overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1)$, $u_0v_3 - u_3v_0 = \mathrm{Im}(\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_3)$, we obtain expression (5.31) for $\boldsymbol{\alpha}_1 = u_1 + v_1\,\mathbf{i}$. Similar arguments for the second and fifth condition in (5.29) yield expression (5.32) for $\boldsymbol{\alpha}_2 = u_2 + v_2\,\mathbf{i}$. From (5.31) and (5.32) we take

$$\mathrm{Im}(\overline{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_2) = \frac{\mathrm{Im}(\overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1)\mathrm{Im}(\overline{\boldsymbol{\beta}}_3\boldsymbol{\beta}_2) - \mathrm{Im}(\overline{\boldsymbol{\beta}}_3\boldsymbol{\beta}_1)\mathrm{Im}(\overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_2)}{\mathrm{Im}(\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_3)}$$

and the numerator of this expression simplifies to give

$$\mathrm{Im}(\overline{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_2) = \frac{\mathrm{Im}(\overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_3)\mathrm{Im}(\overline{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2)}{\mathrm{Im}(\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_3)}. \tag{5.34}$$

Substituting this into the third of equations (5.29) then, after some manipulation, yields condition (5.33). Conversely, let $\mathrm{Im}(\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_3) \neq 0$ and the equations (5.31),(5.32) and (5.33) are satisfied. Multiplying (5.31) and its conjugate by $\overline{\boldsymbol{\alpha}}_0$ and $\boldsymbol{\alpha}_3$ and taking the imaginary part then yields the first and fourth conditions in (5.29). Finally, noting that (5.31),(5.32) and (5.33) imply (5.34), multiplying out condition (5.33), substituting (5.34), and simplifying we take

$$\mathrm{Im}(\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_3)[3\mathrm{Im}(\overline{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_2 + \overline{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2) + \mathrm{Im}(\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_3 + \overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_3)] = 0.$$

Since $\mathrm{Im}(\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_3) \neq 0$, the third condition in (5.29) must hold. ∎

As in mentioned in Section 5.3, the reduction to normal form is used to simplify the study of RRMF curves. Here we achieve simplification of the conditions (5.29) by using the *canonical form*.

Definition 5.1 The PH curve defined by (5.27) and (5.28) is in *canonical form* if $(\boldsymbol{\alpha}_0, \boldsymbol{\beta}_0) = w_i(1,0)$ with $w_i \neq 0$, so that $(\mathbf{e}_1(0), \mathbf{e}_2(0), \mathbf{e}_3(0)) = (\mathbf{i}, \mathbf{j}, \mathbf{k})$

A regular curve, with $\mathbf{r}'(0) \neq \mathbf{0}$, can always be mapped to canonical form through a spatial rotation. Since the assumption of canonical form amounts to the adoption of a special coordinate system, any results we deduce for curves in canonical form must apply to PH curves in general position.

Remark 5.4. The definition of canonical form differs somewhat from prior use [21], where the initial derivative $\mathbf{r}'(0)$ was mapped to the vector $\mathbf{i}$. In Definition 5.1 no scaling is invoked, since the parameter w_i is used to adjust $|\mathbf{r}'(0)| = w_i^2$. Instead, a standard orientation of the normal–plane vectors $\mathbf{e}_2(0), \mathbf{e}_3(0)$ about the tangent $\mathbf{e}_1(0)$ is imposed here.

Lemma 5.6. In canonical form with $(\boldsymbol{\alpha}_0, \boldsymbol{\beta}_0) = w_i(1,0)$ the degree 7 PH curves defined by (5.27) and (5.28) that have rotation-minimizing ERFs are characterized by the conditions

$$\boldsymbol{\alpha}_1 = \frac{\mathrm{Im}(\overline{\boldsymbol{\beta}}_3\boldsymbol{\beta}_1)}{\mathrm{Im}(\boldsymbol{\alpha}_3)}, \quad \boldsymbol{\alpha}_2 = \frac{\mathrm{Im}(\overline{\boldsymbol{\beta}}_3\boldsymbol{\beta}_2)}{\mathrm{Im}(\boldsymbol{\alpha}_3)}, \quad \mathrm{Im}(w_i\boldsymbol{\alpha}_3 + 3\overline{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2) = 0. \tag{5.35}$$

Proof: The expressions for $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ in (5.35) arise on setting $(\boldsymbol{\alpha}_0, \boldsymbol{\beta}_0) = w_i(1,0)$ in (5.31),(5.32) and (5.33). Also after this substitution in (5.33) yields

$$w_i\mathrm{Im}(\boldsymbol{\alpha}_3)\mathrm{Im}(w_i\boldsymbol{\alpha}_3 + 3\overline{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2) = 0,$$

and since $\mathrm{Im}(\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_3) = w_i\mathrm{Im}(\boldsymbol{\alpha}_3) \neq 0$, we obtain the third equation in (5.35). ∎

Recall that, since the reduction to canonical form corresponds to a spatial rotation of the hodograph, it does not affect the satisfaction of (4.15). We focus on canonical-form curves that satisfy (4.15) with $\alpha(t), \beta(t)$ constant, so that $\mathrm{Im}(\overline{\boldsymbol{\alpha}}\boldsymbol{\alpha}' + \overline{\boldsymbol{\beta}}\boldsymbol{\beta}') \equiv 0$. On such curves, characterized by the above Lemma, the ERF is rotation-minimizing, and the normal-plane rotation (4.14) is not required.

Example 5.3.1 With $w_i = 1$, the conditions (5.35) are satisfied by the values

$$\boldsymbol{\alpha}_0 = 1, \quad \boldsymbol{\alpha}_1 = -\frac{5}{3}, \quad \boldsymbol{\alpha}_2 = \frac{8}{3}, \quad \boldsymbol{\alpha}_3 = 2 + 3\,\mathbf{i}$$

$$\boldsymbol{\beta}_0 = 0, \quad \boldsymbol{\beta}_1 = -2 - \mathbf{i}, \quad \boldsymbol{\beta}_2 = 3 + 2\,\mathbf{i}, \quad \boldsymbol{\beta}_3 = 1 - 2\,\mathbf{i}$$

Then the polynomials which defines the Hopf map form of $\mathbf{r}'(t)$ are

$$\begin{aligned}
\boldsymbol{\alpha}(t) &= (1 - 8t + 21t^2 - 12t^3) + (3t^3)\,\mathbf{i} \\
\boldsymbol{\beta}(t) &= (-6t + 21t^2 - 14t^3) + (-3t + 12t^2 - 11t^3)\,\mathbf{i}
\end{aligned}$$

and hence the expression

$$\overline{\boldsymbol{\alpha}}\boldsymbol{\alpha}' + \overline{\boldsymbol{\beta}}\boldsymbol{\beta}' = -8 + 151t - 1026t^2 + 2904t^3 - 3390t^4 + 1410t^5$$

is a real polynomial, as required for a rotation-minimizing ERF. The components $x'(t), y'(t), z'(t)$ of $\mathbf{r}'(t)$ and the parametric speed $\sigma(t)$ are

$$\begin{aligned}
x'(t) &= 1 - 16t + 61t^2 - 36t^3 - 186t^4 + 348t^5 - 164t^6, \\
y'(t) &= -12t + 138t^2 - 616t^3 + 1232t^4 - 1020t^5 + 270t^6, \\
z'(t) &= 6t - 72t^2 + 340t^3 - 788t^4 + 876t^5 - 348t^6, \\
\sigma(t) &= 1 - 16t + 151t^2 - 684t^3 + 1452t^4 - 1356t^5 + 470t^6.
\end{aligned}$$

One can verify that $[\,\mathbf{r}'(t) \times \mathbf{r}''(t)]\,\mathbf{r}'''(t) \neq 0$, for each t and hence $\mathbf{r}'(t)$ is a true space curve.

Remark 5.5. The characterization of canonical–form degree 7 PH curves with rotation-minimizing ERFs in (5.5) assumed that $\mathrm{Im}(\overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_3) \neq 0$. In fact, this condition is necessary for space curve. For brevity, we consider it in the case of canonical-form curves where it becomes $\mathrm{Im}(\boldsymbol{\alpha}_3) \neq 0$. For a canonical-form curve with $\boldsymbol{\alpha}_0 = w_i$ and $\boldsymbol{\beta}_0 = 0$, from the first two conditions in (5.29) we have $w_i\mathrm{Im}(\boldsymbol{\alpha}_1) = w_i\mathrm{Im}(\boldsymbol{\alpha}_2) = 0$, and thus $\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \in \mathbb{R}$. If $\mathrm{Im}(\boldsymbol{\alpha}_3) = 0$, then the remaining three conditions in (5.29) become

$$\mathrm{Im}(\overline{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2) = \mathrm{Im}(\overline{\boldsymbol{\beta}}_1\boldsymbol{\beta}_3) = \mathrm{Im}(\overline{\boldsymbol{\beta}}_2\boldsymbol{\beta}_3) = 0. \tag{5.36}$$

But, since $\overline{\boldsymbol{\beta}}_i\boldsymbol{\beta}_j = \boldsymbol{\beta}_i\boldsymbol{\beta}_j - (\boldsymbol{\beta}_i \times \boldsymbol{\beta}_j)\,\mathbf{j}$ we have that $\mathrm{Im}(\overline{\boldsymbol{\beta}}_1\boldsymbol{\beta}_3) = \overline{\boldsymbol{\beta}}_1 \times \boldsymbol{\beta}_3 = 0$ and thus $\overline{\boldsymbol{\beta}}_1, \boldsymbol{\beta}_3$ are linearly dependent (denoted by $//$). Similarly, from the two remainder conditions of (5.36) we obtain that $\overline{\boldsymbol{\beta}}_1//\boldsymbol{\beta}_2$ and $\overline{\boldsymbol{\beta}}_2//\boldsymbol{\beta}_3$. Obviously, $\boldsymbol{\beta}_2//\boldsymbol{\beta}_3$. Note that from $\mathrm{Im}(\overline{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2) = 0 = -\mathrm{Im}(\overline{\boldsymbol{\beta}}_2\boldsymbol{\beta}_1)$ yields $\boldsymbol{\beta}_1//\overline{\boldsymbol{\beta}}_2$ and finally $\boldsymbol{\beta}_1//\boldsymbol{\beta}_2//\boldsymbol{\beta}_3$. Therefore, $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3$ are of the form $l_1\mathrm{exp}(\,\mathbf{i}\psi)$, $l_2\mathrm{exp}(\,\mathbf{i}\psi)$, $l_3\mathrm{exp}(\,\mathbf{i}\psi)$. Then, since $\boldsymbol{\alpha}(t)$ is real and $\boldsymbol{\beta}(t) = [l_1b_1^3(t) + l_2b_2^3(t) + l_3b_3^3(t)]\mathrm{exp}(\,\mathbf{i}\psi)$, we see that $\mathbf{r}'(t)$ as defined by (4.8) lies in the plane through the origin with unit normal $\mathbf{n} = (0, \sin\psi, -\cos\psi)$.

5.3.2 Necessary and sufficient conditions in terms of quaternion form

Let the polynomial $\mathcal{A}(t) = u(t) + \mathbf{i}\,v(t) + \mathbf{j}\,p(t) + \mathbf{k}\,q(t)$ defines a PH curve of degree 7. Now our focus is to find necessary and sufficient conditions in terms of the coefficients of the polynomials $u(t), v(t), p(t), q(t)$ in order for the PH curve of degree 7 to have the ERF as RMF. Furthermore, we shall give a parametrization of the set of these curves.

Like before, here we use the normal form of the real polynomials $u(t), v(t)$, $p(t), q(t)$ in order to simplify the analysis without any influence in the intrinsic nature of a spatial PH curve. Thus, by using Lemma 4.3, we can assume that

$$u(t) = t^3 + u_2t^2 + u_1t + u_0, \qquad v(t) = v_2t^2 + v_1t + v_0,$$

$$p(t) = p_2t^2 + p_1t + p_0, \qquad q(t) = q_2t^2 + q_1t + q_0.$$

define a PH curve of degree 7 through (4.5). Our purpose is to characterize and parametrize - in terms of the coefficients of the above polynomials - the set V of the polynomials which define RRMF curves of type $(3, 0)$, i.e., the curves of 7 degree which satisfy (4.18) or equivalently

$$uv' - u'v - pq' + p'q = 0. \tag{5.37}$$

In the following we shall identify the space V with the set of the RRMF curves of type $(3, 0)$. After substituting $u(t), v(t), p(t), q(t)$ to (5.37) we equivalently obtain

$$
\begin{aligned}
v_1 &= 0,\\
v_2 &= 0,\\
u_2 v_0 + p_0 q_2 - p_2 q_0 &= 0,\\
u_1 v_0 + p_0 q_1 - p_1 q_0 &= 0,\\
3v_0 + p_1 q_2 - p_2 q_1 &= 0. \qquad (5.38)
\end{aligned}
$$

These relations are the sufficient and necessary conditions for PH curves of degree 7 for which the ERF is RMF. Moreover, by solving the system of the last three equation of (5.38) with unknowns p_0, q_0, v_0, we have the following parametrization of V:

$$
\begin{aligned}
V \;=\; &\{v_1 = v_2 = 0, p_0 = \frac{1}{3}p_1 u_2 - \frac{1}{3}p_2 u_1, q_0 = \frac{1}{3}u_2 q_1 - \frac{1}{3}u_1 q_2,\\
&v_0 = \frac{1}{3}p_2 q_1 - \frac{1}{3}p_1 q_2, \text{ where } p_1, p_2, q_1, q_2, u_0, u_1, u_2 \in \mathbb{R}\}.
\end{aligned}
$$

Since $V = V_l \cup V_{tp} \cup V_{ts}$, where V_l, V_{tp}, V_{ts} are the subset of straight lines, true planar and true spatial curves respectively, below we shall distinct the sets V_l and V_{tp}.

From Proposition 4.1, a curve $\mathbf{r}(t)$ of V is a planar curve other than a straight line if and only if condition (4.24) is satisfied with $(p(t), q(t)) \neq (0,0)$. Combining (5.38) and (4.24) we have

$$
v(t) = 0, \quad p_0 q_2 - p_2 q_0 = 0, \quad p_0 q_1 - p_1 q_0 = 0, \quad p_1 q_2 - p_2 q_1 = 0, \qquad (5.39)
$$

where at least one of $p_0, p_1, p_2, q_0, q_1, q_2$ is non–zero. The last conditions are equivalent to

$$
v(t) = 0 \;\text{ and }\; a\,p(t) + b\,q(t) = 0, \;\text{ with }\; (a,b) \neq (0,0) \;\text{ and }\; (p(t), q(t)) \neq (0,0). \qquad (5.40)
$$

But in view of (5.38) and (5.40) we have that

$$
v(t) = 0 \;\text{ if and only if }\; a\,p(t) + b\,q(t) = 0
$$

.

Now, in order to characterize the set V_l of straight lines we require conditions (5.38) to be satisfied with $p(t) = q(t) = 0$. Thus, we easily see that the set V_l is defined by

$$v(t) = p(t) = q(t) = 0. \tag{5.41}$$

Summarizing the previous results, a planar (included a straight line) PH curve of degree 7 is of type $(3, 0)$ if and only if $v(t) = 0$. Then, the set of true spatial curves V_{ts} consists of the curves which satisfy (5.38) with $v(t) \neq 0$, i.e., $v_0 \neq 0$. Hence, from (5.38) and from $v_0 \neq 0$ we deduce that one parameterization of V_{ts} is given by

$$v_1 = 0, \quad v_2 = 0, \quad v_0 = \frac{p_2q_1 - p_1q_2}{3}, \quad u_2 = 3\frac{p_0q_2 - p_2q_0}{p_2q_1 - p_1q_2}, \quad u_1 = 3\frac{p_0q_1 - p_1q_0}{p_2q_1 - p_1q_2}, \tag{5.42}$$

where $u_0, p_0, p_1, p_2, q_0, q_1, q_2$ are free real parameters.

Example 5.3.2 Choosing the values $u_0 = 0,\ p_0 = 1,\ p_1 = -1,\ p_2 = 1,\ q_0 = 0,\ q_1 = 2,\ q_2 = 1$ in (5.42) gives

$$v_0 = 1,\ u_1 = \frac{2}{3},\ u_1 = \frac{1}{3},$$

and thus we obtain

$$u(t) = t^3 - t^2 + \frac{2}{3},\ v(t) = 1,\ p(t) = t^2 - t + 1,\ q(t) = t^2 + 2t$$

which satisfy (5.37).

The resulting hodograph components

$$\begin{aligned} x'(t) &= t^6 - 2t^5 + \frac{1}{3}t^4 - \frac{10}{3}t^3 - \frac{59}{9}t^2 + 2t, \\ y'(t) &= 2t^5 + 2t^4 - \frac{8}{3}t^3 + \frac{14}{3}t^2 - 2t + 2, \\ z'(t) &= 2t^5 - 4t^4 + \frac{16}{3}t^3 - \frac{10}{3}t^2 - 4t, \end{aligned}$$

define a primitive curve with $\gcd(x'(t), y'(t), z'(t)) = 1$ and satisfy

$$x'^2(t) + y'^2(t) + z'^2(t) = \sigma^2(t),$$

where

$$\sigma(t) = t^6 + 2t^5 - \frac{5}{3}t^4 - \frac{2}{3}t^3 - \frac{47}{9}t^2 - 4t + 2$$

The hodograph defines a true space curve, as can be verified from the fact that $v_0 \neq 0$.

5.3.3 Necessary and sufficient conditions in terms of factorization

Now we consider the quaternion polynomial $\mathcal{A}(t)$ in factorization form and we are interested in finding necessary and sufficient conditions in terms of its factorization in order for the curve to be of type $(3,0)$.

Proposition 5.6. *Let $\mathcal{A}(t) = (t - \mathcal{C}_1)(t - \mathcal{C}_2)(t - \mathcal{C}_3)$ be a quaternion polynomial that defines a PH curve $\mathbf{r}(t)$ of degree 7. Set $S_1 = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3$, $S_2 = \mathcal{C}_1\mathcal{C}_3 + \mathcal{C}_2\mathcal{C}_3 + \mathcal{C}_1\mathcal{C}_2$ and $S_3 = \mathcal{C}_1\mathcal{C}_2\mathcal{C}_3$. The curve has a rotation minimizing ERF if and only if the following conditions hold*

$$\mathbf{i} \cdot \mathrm{vect}(S_1) = \mathbf{i} \cdot \mathrm{vect}(S_2) = 0$$

$$-3\mathbf{i} \cdot \mathrm{vect}(S_3) = \mathrm{vect}(S_2)[\mathrm{vect}(S_1) \times \mathbf{i})]$$

$$\mathbf{i} \cdot \mathrm{vect}(S_3)\ \mathrm{scal}(S_1) = \mathrm{vect}(S_1)[\mathrm{vect}(S_3) \times \mathbf{i})]$$

$$[\mathrm{vect}(S_2) \times\ \mathbf{i}]\,[\mathrm{scal}(S_2)\mathrm{vect}(S_1) + 3\mathrm{vect}(S_3)] = 0$$

Proof: The polynomial $\mathcal{A}(t)$ defines a curve whose ERF is RMF if and only if

$$\mathrm{scal}(\mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}'^{*}(t)) = 0. \tag{5.43}$$

The polynomial $\mathcal{A}(t) = (t - \mathcal{C}_1)(t - \mathcal{C}_2)(t - \mathcal{C}_3)$ takes the form

$$\mathcal{A}(t) = t^3 - S_1 t^2 + S_2 t - S_3.$$

We substitute the quaternion coefficients expressed in the scalar-vector form $\mathcal{C}_i = (c_i, \mathbf{c}_i)$, $i = 1, 2, 3$ in the last form of $\mathcal{A}(t)$ and we have

$$\begin{aligned}\mathcal{A}(t) &= (t^3 - \mathrm{scal}(S_1)\,t^2 + \mathrm{scal}(S_2)\,t - \mathrm{scal}(S_3), \\ &\quad -\mathrm{vect}(S_1)\,t^2 + \mathrm{vect}(S_2)\,t - \mathrm{vect}(S_3))\end{aligned}$$

The conjugate of the derivative is

$$\mathcal{A}'^{*}(t) = (3\,t^2 - 2\,\mathrm{scal}(S_1)\,t + \mathrm{scal}(S_2), 2\,\mathrm{vect}(S_1)\,t - \mathrm{vect}(S_2)).$$

Substituting the $\mathcal{A}(t)$ and $\mathcal{A}'^{*}(t)$ to (5.43) we equivalently get the above conditions. ∎

Chapter 6

Non–primitive hodographs

In this chapter we present the condition under which a PH curve generated by a primitive quaternion polynomial $\mathcal{A}(t)$ has non–primitive hodograph and we characterize such regular curves in terms of its associated polynomial $\mathcal{A}(t)$. We also consider the problem of the generation of RRMF curves from others of lower degree. In Section 6.1 we give necessary and sufficient conditions for such curves to be generated by another of lower degree and some of their geometrical properties are studied as well. In Section 6.2 we focus on some special cases of RRMF curves of degree 5 and 7 and we present the conditions under which these classes of curves have non–primitive hodographs. Recall that throughout this chapter we shall assume that the polynomial $\mathcal{A}(t)$ is primitive.

6.1 Characterization of non–primitive hodographs

Let $\mathcal{A}(t) = \mathbf{f}(t)+\mathbf{k}\,\mathbf{g}(t)$, where $\mathbf{f}(t), \mathbf{g}(t) \in \mathbb{C}[t]$, be a monic primitive quaternion polynomial of degree m and $\mathbf{r}'(t) = \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t)$ be the hodograph generated by $\mathcal{A}(t)$. Recall that the hodograph $\mathbf{r}'(t) = (x'(t), y'(t), z'(t))$ of a PH curve $\mathbf{r}(t)$ is primitive if $\gcd(x'(t), y'(t), z'(t)) = 1$. As it is mentioned in Chapter 5, it is possible for a PH curve to be regular even when its hodograph $\mathbf{r}'(t)$ is non–primitive.

Definition 6.1 A quaternion polynomial $\mathcal{A}(t) = u(t)+\mathbf{i}\,v(t)+\mathbf{j}\,p(t)+\mathbf{k}\,q(t)$ of degree ≥ 1 (not necessarily primitive) is said to be *right (left) reducible* over $\mathbb{C}$ if a complex polynomial $\mathbf{c}(t) \in \mathbb{C}[t]$ of degree ≥ 1 exists, such that

$$\mathcal{A}(t) = \mathcal{B}(t)\,\mathbf{c}(t) \quad (\mathcal{A}(t) = \mathbf{c}(t)\,\mathcal{B}(t)),$$

respectively, for some $\mathcal{B}(t) \in \mathbb{H}[t]$. If $\mathcal{A}(t)$ is not right (left) reducible, it is said to be *right (left) irreducible* over $\mathbb{C}$, respectively.

Remark 6.1. It is easily seen that if a quaternion polynomial $\mathcal{A}(t)$ is primitive and right irreducible over $\mathbb{C}$ then every right root of $\mathcal{A}(t)$ belongs to $\mathbb{H} \setminus \mathbb{C}$.

In the following by "reducible" ("irreducible") we shall mean "right reducible" ("right irreducible"), respectively.

The next theorem characterizes the non–primitives hodographs.

Theorem 6.1. *The following statements are equivalent:*
(a) The hodograph defined by (4.5) *is non–primitive.*
(b) $\mathcal{A}(t) = \mathcal{B}(t)\,\mathcal{C}(t)$, *where* $\mathcal{B}(t) \in \mathbb{H}[t]$ *and* $\mathcal{C}(t)$ *is a non–constant polynomial of* $\mathbb{C}[t]$.
(c) $\mathbf{r}'(t) = f(t)\mathcal{B}(t)\,\mathbf{i}\,\mathcal{B}^*(t)$, *where* $f(t)$ *is a real monic polynomial with no real roots and* $\mathcal{B}(t)$ *is a left factor of* $\mathcal{A}(t)$.
(d) $\mathit{Resultant}_t(\mathbf{f}(t), \mathbf{g}(t)) = 0$.

Proof: Suppose that the hodograph $\mathbf{r}'(t)$ is not primitive. Then, there is a real monic irreducible polynomial $\wp(t)$ which divides the polynomials

$$u^2(t) + v^2(t) - p^2(t) - q^2(t), \quad u(t)q(t) + v(t)p(t), \quad v(t)q(t) - u(t)p(t).$$

The relations $\wp(t)|u(t)q(t) + v(t)p(t)$ and $\wp(t)|v(t)q(t) - u(t)p(t)$ imply

$$\wp(t)|u^2(t)q(t) + u(t)v(t)p(t) \quad \text{and} \quad \wp(t)|v^2(t)q(t) - u(t)v(t)p(t)$$

and whence we get $\wp(t)|(u^2(t) + v^2(t))q(t)$.

Suppose first that $\wp(t) \not| q(t)$. Then, we have $\wp(t)|u^2(t) + v^2(t)$, and so, the relation $\wp(t)|u^2(t) + v^2(t) - p^2(t) - q^2(t)$ yields $\wp(t)|p^2(t) + q^2(t)$. If $\wp(t) = t - a$, then the real number a is a root of $p^2(t) + q^2(t)$, and so a common root of $p(t)$ and $q(t)$. Similarly, a is a common root of $u(t)$ and $v(t)$. Thus, we have $\gcd(u(t), v(t), p(t), q(t)) > 1$ which is a contradiction. Therefore, we have $\deg \wp(t) = 2$.

Next, suppose that $\wp(t)|q(t)$. Thus, we have

$$\wp|u^2(t) + v^2(t) - p^2(t), \quad \wp(t)|v(t)p(t), \quad \wp(t)|u(t)p(t).$$

If $\wp(t) \not| p(t)$, then $\wp(t)|v(t)$, $\wp(t)|u(t)$ and the relation $\wp(t)|u^2(t) + v^2(t) - p^2(t)$ implies that $\wp(t)|p(t)$, which is a contradiction. Therefore $\wp(t)|p(t)$ and so, we have $\wp(t)|p^2(t) + q^2(t)$, whence follows that $\wp(t)|p^2(t) + q^2(t)$. If $\wp(t) = t-a$, a is a common root of $u(t)$ and $v(t)$, and since $t-a|p(t)$, $t-a|q(t)$ we have $\gcd(u(t), v(t), p(t), q(t)) > 1$, which is a contradiction. Hence, we have $\deg \wp(t) = 2$. Therefore, in both cases, we have that $\wp(t)|u^2(t)+v^2(t)$, $\wp(t)|p^2(t) + q^2(t)$ and $\wp(t) = (t - \mathbf{r})\,(t - \bar{\mathbf{r}})$, where $\mathbf{r} \in \mathbb{C} \setminus \mathbb{R}$ and $\bar{\mathbf{r}}$ is the complex conjugate of $\mathbf{r}$.

We remark that the divisibility relations

$$\wp(t)|u(t)q(t) + v(t)p(t) \quad \text{and} \quad \wp(t)|v(t)q(t) - u(t)p(t)$$

can be equivalently presented by the relation

$$(t - \mathbf{r})(t - \bar{\mathbf{r}})|(u(t) + v(t)\,\mathbf{i})\,(q(t) - p(t)\,\mathbf{i}).$$

We also have

$$(t-\mathbf{r})(t-\bar{\mathbf{r}})|(u(t)+v(t)\,\mathbf{i})\,(u(t)-v(t)\,\mathbf{i}), \quad (t-\mathbf{r})(t-\bar{\mathbf{r}})|(q(t)+p(t)\,\mathbf{i})\,(q(t)-p(t)\,\mathbf{i}).$$

Suppose that $t - \mathbf{r}|u(t) + v(t)\,\mathbf{i}$. If $t - \mathbf{r} \not| q(t) + p(t)\,\mathbf{i}$, then $t - \mathbf{r}|q(t) - p(t)\,\mathbf{i}$ and so, $t - \bar{\mathbf{r}}|q(t) + p(t)\,\mathbf{i}$. On the other hand, we have that the relation $t - \mathbf{r} \not| q(t) + p(t)\,\mathbf{i}$ implies $t - \bar{\mathbf{r}} \not| q(t) - p(t)\,\mathbf{i}$. Thus, the relation $(t - \mathbf{r})(t - \bar{\mathbf{r}})|(u(t)+v(t)\,\mathbf{i})(q(t)-p(t)\,\mathbf{i})$ implies that $t-\bar{\mathbf{r}}|u(t)+v(t)\,\mathbf{i}$. Thus, we deduce that the polynomials $u(t) + v(t)\,\mathbf{i}$ and $q(t) + p(t)\,\mathbf{i}$ have a common complex root. If $t-\mathbf{r}|q(t)+p(t)\,\mathbf{i}$, then we also have that $u(t)+v(t)\,\mathbf{i}$ and $q(t)+p(t)\,\mathbf{i}$ have a common complex root. Since $\mathcal{A}(t) = u(t) + v(t)\,\mathbf{i} + \mathbf{k}\,(q(t) + p(t)\,\mathbf{i})$, it follows that $\mathcal{A}(t)$ has a complex root.

Suppose that $\mathcal{A}(t) = \mathcal{B}(t)\,\mathcal{C}(t)$, where $\mathcal{B}(t) \in \mathbb{H}[t]$ and $\mathcal{C}(t)$ is a monic polynomial of $\mathbb{C}[t] \setminus \mathbb{R}[t]$ with $\deg C(t) > 0$. Then we have

$$\mathbf{r}'(t) = f(t)\,\mathcal{B}(t)\,\mathbf{i}\,\mathcal{B}^*(t),$$

where $f(t) = \mathcal{C}(t)\,\mathcal{C}^*(t)$ is a real monic polynomial with non real root. It follows that the hodograph $\mathbf{r}'(t)$ is non–primitive.

Thus, we have established the equivalence of propositions (a), (b) and (c). Finally, Corollary 2.1 provides the equivalence of (b) and (d).

Corollary 6.1. *The hodograph* $\mathbf{r}'(t)$ *is primitive if and only if the quaternion polynomial* $\mathcal{A}(t)$ *has no complex roots.*

Corollaries 2.1 and 2.3 give immediately the following results.

Corollary 6.2. *Suppose that the hodograph* $\mathbf{r}'(t)$ *is generated by the polynomial* $\mathcal{A}(t) = t^2 + \mathcal{B}t + \mathcal{C}$. *Set* $\mathcal{B} = \mathbf{b}_1 + \mathbf{k}\,\mathbf{c}_1$ *and* $\mathcal{C} = \mathbf{b}_0 + \mathbf{k}\,\mathbf{c}_0$, *where* $\mathbf{b}_0, \mathbf{b}_1, \mathbf{c}_0, \mathbf{c}_1 \in \mathbb{C}$. *Then,* $\mathbf{r}'(t)$ *is non–primitive if and only if*

$$\mathbf{c}_0^2 - \mathbf{c}_0\mathbf{b}_1\mathbf{c}_1 + \mathbf{b}_0\mathbf{c}_1^2 = 0.$$

Remark 6.2. From the proof of Theorem 6.1 we also conclude that if the hodograph $\mathbf{r}'(t)$ is non–primitive then the $\gcd(x'(t), y'(t), z'(t))$ has no real roots. In other words, if $\gcd(x'(t), y'(t), z'(t))$ has real root then the polynomial $\mathcal{A}(t)$ is non–primitive.

Remark 6.3. Let $\mathcal{A}(t) = \mathcal{A}_n t^n + \cdots + \mathcal{A}_0$ and set $\tilde{\mathcal{A}}(t) = \mathcal{A}(t)/\mathcal{A}_n$. By Theorem 2.1, we can write $\tilde{\mathcal{A}}(t) = \mathcal{B}(t)\mathbf{c}(t)$, where $\mathbf{c}(t) \in \mathbb{C}[t]$ and $\mathcal{B}(t)$ has no right factor in $\mathbb{C}[t]$. Thus, we have $\mathcal{A}(t) = (\mathcal{A}_n\, B(t))\mathbf{c}(t)$.

Definition 6.2 We call *level of non–primitivity* of the hodograph $\mathbf{r}'(t)$, and we denote by $\ell(\mathbf{r}'(t))$, the maximum of the degrees of complex polynomials $\mathbf{c}(t)$ such that there is a quaternion polynomial $\mathcal{B}(t)$ such that $\mathcal{A}(t) = \mathcal{B}(t)\,\mathbf{c}(t)$, with $\mathcal{B}(t) \in \mathbb{H}[t]$ and $\mathbf{c}(t) \in \mathbb{C}[t]$.

Note that the level of non–primitivity of a primitive hodograph is zero.

Theorem 6.2. *The level of non–primitivity of the hodograph* $\mathbf{r}'(t)$ *generated by the polynomial* $\mathcal{A}(t)$ *is*

$$\ell(\mathbf{r}'(t)) = \deg \mathcal{A} - \operatorname{rankBez}(\mathbf{f}, \mathbf{g}).$$

Since the polynomial $\mathcal{B}(t)$ generates the hodograph $\hat{\mathbf{r}}'(t) = \mathcal{B}(t)\,\mathbf{i}\,\mathcal{B}^*(t)$ we can give the following definition.

Definition 6.3 We say that a polynomial curve $\mathbf{r}(t)$ is *generated* by another polynomial curve $\hat{\mathbf{r}}(t)$, and we write $\mathbf{r}(t) \succeq \hat{\mathbf{r}}(t)$, if the hodograph of $\mathbf{r}(t)$ is a scalar polynomial multiple of $\hat{\mathbf{r}}(t)$ — i.e., $\mathbf{r}'(t) = f(t)\,\hat{\mathbf{r}}'(t)$ for some monic real polynomial $f(t)$ with non real roots. We shall also say that the curve $\hat{\mathbf{r}}(t)$ *generates* the curve $\mathbf{r}(t)$.

Clearly, a PH curve with a non–primitive hodograph is generated by another PH curve, of lower degree. Such curves are defined by quaternion polynomials $\mathcal{A}(t)$ that admit factorizations of the form $\mathcal{A}(t) = \mathcal{B}(t)\,\mathbf{c}(t)$, where $\mathbf{c}(t)$ is a non constant complex polynomial with no real roots. Thus, a PH curve $\mathbf{r}(t)$ is generated by another PH curve of lower degree if and only if the level of non–primitivity of its hodograph $\mathbf{r}'(t)$ is > 0. Theorems 6.2 and 6.1 with Corollary 6.2 give necessary and sufficient condition for it.

Proposition 6.1. *The relation $\succeq$ is a partial ordering on the set of polynomial curves C.*

Proof: For every $\mathbf{r}(t) \in C$ we clearly have $\mathbf{r}(t) \succeq \mathbf{r}(t)$. Suppose that $\mathbf{r}(t) \succeq \hat{\mathbf{r}}(t)$ and $\hat{\mathbf{r}(t)} \succeq \mathbf{r}(t)$. Then there are real monic polynomials $f(t)$ and $g(t)$ with non real roots such that $\mathbf{r}'(t) = f(t)\,\hat{\mathbf{r}}'(t)$ and $\hat{\mathbf{r}}'(t) = g(t)\mathbf{r}'(t)$. Thus, we get $\mathbf{r}'(t) = f(t)\,g(t)\mathbf{r}'(t)$, whence we obtain $f(t) = g(t) = 1$. Hence $\mathbf{r}(t) = \hat{\mathbf{r}}(t)$. Finally, suppose that $\mathbf{r}_1(t) \succeq \mathbf{r}_2(t)$ and $\mathbf{r}_2(t) \succeq \mathbf{r}_3(t)$. It follows that there are real monic polynomials $f_1(t)$ and $f_2(t)$ with non real roots such that $\mathbf{r}_1'(t) = f_1(t)\,\mathbf{r}_2'(t)$ and $\mathbf{r}_2'(t) = f_2(t)\mathbf{r}_3'(t)$. Thus, we have $\mathbf{r}_1'(t) = f_1(t)\,f_2(t)\mathbf{r}_3'(t)$, whence we get $\mathbf{r}_1(t) \succeq \mathbf{r}_3(t)$. Therefore, the relation $\succeq$ is reflexive, antisymmetric and transitive and so is a partial ordering on C. ∎

Remark 6.4. The polynomial curves having primitive hodograph are the minimal elements of this ordering.

6.1.1 Geometrical properties

Here we discuss the geometrical interpretation of a PH curve which is generated by another PH curve of lower degree. More precisely, we are interested in finding the relation between these curves in the space and if the geometrical properties of the one are transfered to the other. Also we wish to know how these curves are represented and what relation do their graphs have.

Let $\mathbf{r}(t)$ be a PH curve defined by the quaternion polynomial

$$\mathcal{A}(t) = u(t) + \mathbf{i}\,v(t) + \mathbf{j}\,p(t) + \mathbf{k}\,q(t)$$

and $\hat{\mathbf{r}}(t)$ be a PH curve of lower degree than $\mathbf{r}(t)$, defined by the quaternion polynomial

$$\mathcal{B}(t) = \hat{u}(t) + \mathbf{i}\,\hat{v}(t) + \mathbf{j}\,\hat{p}(t) + \mathbf{k}\,\hat{q}(t)$$

such that

$$\mathbf{r}'(t) = f(t)\,\mathcal{B}(t)\,\mathbf{i}\,\mathcal{B}^*(t),$$

for some monic real polynomial $f(t)$ with no real roots. We assume that $(\mathbf{t}, \mathbf{h}, \mathbf{b})$, $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, $\kappa_1(t)$, $\tau_1(t)$, $\sigma_1(t)$ and $(\hat{\mathbf{t}}, \hat{\mathbf{h}}, \hat{\mathbf{b}})$, $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$, $\kappa_2(t)$, $\tau_2(t)$, $\sigma_2(t)$ are the Frenet frame, Euler–Rodrigues frame, curvature, torsion and parametric speed of $\mathbf{r}(t)$, $\hat{\mathbf{r}}(t)$ at each t, respectively.

It can be easily verified, that if the curve $\mathbf{r}(t)$ is generated by the lower–order curve $\hat{\mathbf{r}}(t)$, then it has, at each t, the same Frenet and Euler–Rodrigues frame as the latter. Also the parametric speed of $\mathbf{r}(t)$ is equal to that of $\hat{\mathbf{r}}(t)$ multiplied by $|f(t)|$, while the curvature and torsion of $\mathbf{r}(t)$ equal those of $\hat{\mathbf{r}}(t)$ divided by $|f(t)|$ and $f(t)$, respectively.

Indeed, if we substitute $\mathbf{r}' = f\,\hat{\mathbf{r}}'$ and its derivatives into the definitions of the tangent, principal normal, and binormal,

$$\mathbf{t} = \frac{\mathbf{r}'}{|\mathbf{r}'|}, \quad \mathbf{h} = \frac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}' \times \mathbf{r}''|} \times \mathbf{t}, \quad \mathbf{b} = \frac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}' \times \mathbf{r}''|},$$

and the parametric speed, curvature, and torsion,

$$\sigma = |\mathbf{r}'|, \quad \kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}, \quad \tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}.$$

we obtain that

$$\mathbf{t} = \hat{\mathbf{t}}, \quad \mathbf{h} = \hat{\mathbf{h}}, \quad \mathbf{b} = \hat{\mathbf{b}}$$

and

$$\sigma_1(t) = |f(t)|\,\sigma_2(t), \quad \frac{\kappa_1}{\kappa_2} = \frac{1}{|f(t)|}, \quad \frac{\tau_1}{\tau_2} = \frac{1}{f(t)}, \quad \text{for each } t.$$

By Theorem 6.1 we have that that $f(t) > 0$ and so $|f(t)| = f(t)$, $\forall\ t \in \mathbb{R}$. Thus, we may write that for the PH curves $\mathbf{r}(t)$ and $\hat{\mathbf{r}}(t)$ we have

$$\frac{\kappa_1}{\kappa_2} = \frac{\tau_1}{\tau_2},$$

for the corresponding points of $\mathbf{r}(t)$ and $\hat{\mathbf{r}}(t)$.

Now we shall prove that $\mathbf{e}_1 = \hat{\mathbf{e}}_1$, $\mathbf{e}_2 = \hat{\mathbf{e}}_2$, $\mathbf{e}_3 = \hat{\mathbf{e}}_3$, at each t. Indeed, since $\mathbf{r}'(t) = f(t)\hat{\mathbf{r}}'(t)$ we have that

$$\mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t) = f(t)\mathcal{B}(t)\,\mathbf{i}\,\mathcal{B}^*(t), \tag{6.1}$$

which is equivalently to

$$\begin{aligned} u^2(t)+v^2(t)-p^2(t)-q^2(t) &= f(t)\,[\hat{u}^2(t)+\hat{v}^2(t)-\hat{p}^2(t)-\hat{q}^2(t)],\\ u(t)q(t)+v(t)p(t) &= f(t)\,[\hat{u}(t)\hat{q}(t)+\hat{v}(t)\hat{p}(t)],\\ v(t)q(t)-u(t)p(t) &= f(t)\,[\hat{v}(t)\hat{q}(t)-\hat{u}(t)\hat{p}(t)]. \end{aligned} \tag{6.2}$$

The above relations can be otherwise expressed in the form

$$\begin{aligned} \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t) &= f(t)\,\mathcal{B}(t)\,\mathbf{i}\,\mathcal{B}^*(t),\\ \mathcal{A}(t)\,\mathbf{j}\,\mathcal{A}^*(t) &= f(t)\,\mathcal{B}(t)\,\mathbf{j}\,\mathcal{B}^*(t),\\ \mathcal{A}(t)\,\mathbf{k}\,\mathcal{A}^*(t) &= f(t)\,\mathcal{B}(t)\,\mathbf{k}\,\mathcal{B}^*(t). \end{aligned} \tag{6.3}$$

By multiplying (6.1) on the right with $\mathcal{B}(t)$ yields

$$|\mathcal{A}(t)|^2\,\mathbf{i}\,\mathcal{A}^*(t)\,\mathcal{B}(t) = f(t)\,|\mathcal{B}(t)|^2\,\mathcal{A}^*(t)\,\mathcal{B}(t)\,\mathbf{i}$$

which implies that

$$|\mathcal{A}(t)|^2\,|\mathbf{i}\,\mathcal{A}^*(t)\,\mathcal{B}(t)| = |f(t)\,||\mathcal{B}(t)|^2\,|\mathcal{A}^*(t)\,\mathcal{B}(t)\,\mathbf{i}|. \tag{6.4}$$

Since $f(t) > 0$ and $|\mathbf{i}\,\mathcal{A}^*(t)\,\mathcal{B}(t)| = |\mathcal{A}^*(t)\,\mathcal{B}(t)\,\mathbf{i}|$ the (6.4) gives

$$|\mathcal{A}(t)|^2 = f(t)\,|\mathcal{B}(t)|^2 \tag{6.5}$$

Substituting (6.3) and (6.5) into (4.12) and simplifying we deduce that

$$\mathbf{e}_1 = \hat{\mathbf{e}}_1, \quad \mathbf{e}_2 = \hat{\mathbf{e}}_2 \quad \text{and} \quad \mathbf{e}_3 = \hat{\mathbf{e}}_3, \quad \text{at each } t.$$

In the following we shall prove that if the $\mathbf{r}(t)$ is a planar curve (other than a straight line) then $\hat{\mathbf{r}}(t)$ is a planar curve and conversely. By

$$\begin{aligned} \mathcal{A}(t) &= \mathcal{B}(t)(c_1(t)+\mathbf{i}\,c_2(t))\\ &= (\hat{u}(t)+\mathbf{i}\,\hat{v}(t)+\mathbf{j}\,\hat{p}(t)+\mathbf{k}\,\hat{q}(t))(c_1(t)+\mathbf{i}\,c_2(t))\\ &= (\hat{u}(t)+\mathbf{i}\,\hat{v}(t))(c_1(t)+\mathbf{i}\,c_2(t))+(\mathbf{j}\,\hat{p}(t)+\mathbf{k}\,\hat{q}(t))(c_1(t)+\mathbf{i}\,c_2(t))\\ &= U(t)+\mathbf{i}\,V(t)+\mathbf{j}\,P(t)+\mathbf{k}\,Q(t) \end{aligned} \tag{6.6}$$

we have

$$\begin{aligned} U(t) + \mathbf{i}\,V(t) &= (\hat{u}(t) + \mathbf{i}\,\hat{v}(t))(c_1(t) + \mathbf{i}\,c_2(t)) \\ \mathbf{j}\,P(t) + \mathbf{k}\,Q(t) &= (\mathbf{j}\,\hat{p}(t) + \mathbf{k}\,\hat{q}(t))(c_1(t) + \mathbf{i}\,c_2(t)) \end{aligned}$$

and using Lemma 4.1, the last relations imply

$$[\,U(t), V(t)\,] = [\,\hat{u}(t), \hat{v}(t)] + [\,c_1(t), c_2(t)\,] \tag{6.7}$$

and

$$[\,Q(t), P(t)\,] = [\,\hat{q}(t), \hat{p}(t)] + [\,c_1(t), c_2(t)\,] \tag{6.8}$$

Suppose that $\hat{\mathbf{r}}(t)$ is planar, then by (4.24) we have that

$$\frac{\hat{u}(t)\hat{v}'(t) - \hat{u}'(t)\hat{v}(t)}{\hat{u}^2(t) + \hat{v}^2(t)} = \frac{\hat{q}(t)\hat{p}'(t) - \hat{q}'(t)\hat{p}(t)}{\hat{q}^2(t) + \hat{p}^2(t)}$$

which can be written as

$$[\,\hat{u}(t), \hat{v}(t)] = [\,\hat{q}(t), \hat{p}(t)] \tag{6.9}$$

By substituting (6.9) into (6.7) and (6.8) we take

$$[\,U(t), V(t)\,] = [\,Q(t), P(t)\,], \tag{6.10}$$

and thus $\mathbf{r}(t)$ is planar curve as well. Conversely, let $\mathbf{r}(t)$ is planar. Then by (6.10), (6.7) and (6.8) yields (6.9), i.e., $\hat{\mathbf{r}}(t)$ is planar curve.

Now, if $\hat{\mathbf{r}}(t)$ is straight line, we substitute $p(t) = q(t) = 0$ into (6.6) and we take $P(t) = Q(t) = 0$ which means that $\mathbf{r}(t)$ is a straight line too. The converse is obvious by using an analogous argument.

Up to now, we have proved that $\mathbf{r}(t)$ is a planar curve if and only if $\hat{\mathbf{r}}(t)$ is a planar curve and $\mathbf{r}(t)$ is a straight line if and only if $\hat{r}(t)$ is a straight line. Consequently, if $\mathbf{r}(t)$ is a (true) space curve then $\hat{\mathbf{r}}(t)$ is a (true) space curve and conversely.

Concerning the RRMF condition, one can verify that $\mathbf{r}(t)$ is RRMF curve if and only if $\hat{\mathbf{r}}(t)$ is RRMF curve. Indeed, suppose that $\hat{\mathbf{r}}(t)$ is RRMF. Then there exists real polynomials $b_1(t), b_2(t)$ with $\gcd(b_1(t), b_2(t)) = 1$ such that $[\,\mathcal{B}(t)\,] = [\,b_1(t), b_2(t)\,]$. By

$$\mathcal{A}(t) \;=\; \mathcal{B}(t)(c_1(t) + c_2(t)\,\mathbf{i})\,,$$

implies

$$[\,\mathcal{A}(t)\,] = [\,\mathcal{B}(t)\,] + [\,c_1(t), c_2(t)\,] \tag{6.11}$$

and hence

$$[\,\mathcal{A}(t)\,] = [\,b_1(t), b_2(t)\,] + [\,c_1(t), c_2(t)\,].$$

From Lemma 4.1, $[\,b_1(t), b_2(t)\,] + [\,c_1(t), c_2(t)\,] = [\,a_1(t), a_2(t)\,]$ where $a_1(t) + \mathbf{i}\,a_2(t) = (b_1(t) + \mathbf{i}\,b_2(t))(c_1(t) + \mathbf{i}\,c_2(t))$ and thus $[\,\mathcal{A}(t)\,] = [\,a_1(t), a_2(t)\,]$, i.e., $\mathbf{r}(t)$ is an RRMF curve. Suppose now that $\mathbf{r}(t)$ is an RRMF curve, i.e., there exists $d_1(t), d_2(t) \in \mathbb{R}[t]$ such that $[\,\mathcal{A}(t)\,] = [\,d_1(t), d_2(t)\,]$. By (6.11) we have

$$[\,\mathcal{B}(t)\,] = [\,d_1(t), d_2(t)\,] - [\,c_1(t), c_2(t)\,].$$

Again, in view of Lemma 4.1

$$[\,\mathcal{B}(t)\,] = [\,f_1(t), f_2(t)\,]$$

where $f_1(t) + \mathbf{i}\,f_2(t) = (d_1(t) + \mathbf{i}\,d_2(t))(c_1(t) - \mathbf{i}\,c_2(t))$, and thus $\hat{\mathbf{r}}(t)$ is also an RRMF curve.

The above discussion is summarized as follows.

Proposition 6.2. *Let* $\mathbf{r}(t)$, $\hat{\mathbf{r}}(t)$ *be PH curves with hodograph defined by the quaternion polynomials* $\mathcal{A}(t) = u(t) + \mathbf{i}\,v(t) + \mathbf{j}\,p(t) + \mathbf{k}\,q(t)$ *and* $\mathcal{B}(t) = \hat{u}(t) + \mathbf{i}\,\hat{v}(t) + \mathbf{j}\,\hat{p}(t) + \mathbf{k}\,\hat{q}(t)$*, respectively, such that* $\mathcal{A}(t) = \mathcal{B}(t)\,\mathcal{C}(t)$ *with* $\mathcal{C}(t) \in \mathbb{C}[t]$*. Then*

1. $\mathbf{r}(t)$, $\hat{\mathbf{r}}(t)$ *have the same Frenet and Euler–Rodrigues frames at each* t*.*

2. $\mathbf{r}(t)$ *is planar, straight line and true space curve if and only if* $\hat{\mathbf{r}}(t)$ *is likewise, respectively.*

3. $\mathbf{r}(t)$ *is an RRMF curve if and only if* $\hat{\mathbf{r}}(t)$ *is RRMF.*

4. $$\kappa_1(t)\tau_2(t) = \kappa_2(t)\tau_1(t)$$

   where $\kappa_1(t)$, $\kappa_2(t)$ *and* $\tau_1(t)$ $\tau_2(t)$ *are the curvature and torsion of* $\mathbf{r}(t)$, $\hat{\mathbf{r}}(t)$ *respectively.*

6.2 Non–primitive hodographs of RRMF curves of degree 5 and 7

In this section we study RRMF curves of degree 5 and 7 whose hodographs are non–primitive. More precisely, we deal with the special cases of the RRMF curves of types $(2,1)$, $(2,0)$ and $(3,0)$. We, especially, are interested in the last two types, since these curves possess the specific geometrical property of having the ERF as an RMF and the curves of type $(2,1)$ are those which -as we shall see- generate RRMF curves of type $(3,0)$.

6.2.1 Curves of type $(2,1)$ and $(2,0)$ with non–primitive hodographs

The simplest example of two PH curves $\mathbf{r}(t)$, $\hat{\mathbf{r}}(t)$ such that $\mathbf{r}(t) \succeq \hat{\mathbf{r}}(t)$ concerns the case of a quintic $\mathbf{r}(t)$ defined by a quadratic quaternion polynomial $\mathcal{A}(t)$ that admits a factorization of the form

$$\mathcal{A}(t) \;=\; \mathcal{B}(t)(c_1(t) + c_2(t)\,\mathbf{i})\,.$$

The PH quintic $\mathbf{r}(t)$ is generated by the PH cubic $\hat{\mathbf{r}}(t)$ defined by the hodograph

$$\hat{\mathbf{r}}'(t) \;=\; \mathcal{B}(t)\,\mathbf{i}\,\mathcal{B}^*(t)\,.$$

We shall give necessary and sufficient conditions under which an RRMF quintic with non–primitive hodograph is of type $(2,1)$. Moreover, we prove that there not exists RRMF curves of type $(2,0)$ with non–primitive hodograph.

Consider the quaternion polynomial

$$\mathcal{A}(t) = \boldsymbol{\alpha}(t) + \mathbf{k}\,\boldsymbol{\beta}(t), \tag{6.12}$$

where $\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)$ are considered be in canonical form i. e., $\boldsymbol{\alpha}(t)$ is a monic quadratic polynomial and $\boldsymbol{\beta}(t)$ a linear polynomial. We assume that $\mathcal{A}(t)$ has one no real root. Then

$$\boldsymbol{\alpha}(t) = (t - \mathbf{z}_1)(t - \mathbf{z}_2) \quad \text{and} \quad \boldsymbol{\beta}(t) = \mathbf{c}(t - \mathbf{z}_2) \text{ with } \mathbf{c} \in \mathbb{C}.$$

Proposition 6.3. *The above polynomial $\mathcal{A}(t)$ defines a non–primitive hodograph of an RRMF curve of type $(2,1)$ if and only if $Im(\mathbf{z}_1) = 0$. In this case, the polynomial $\mathbf{w}(t) = a(t) + \mathbf{i}b(t)$, with $a(t) = t - Re(\mathbf{z}_2)$ and $b(t) = -Im(\mathbf{z}_2)$, satisfies condition (4.16). Furthermore, $\mathcal{A}(t)$ does not define an RRMF curve of type $(2,0)$.*

Proof: The polynomial $\mathcal{A}(t)$ defines a hodograph of RRMF curve of type $(2,1)$ if and only if there exists a complex polynomial $\mathbf{w}(t) = a(t) + \mathbf{i}\, b(t)$ with $a(t) = t - a_0$, $b(t) = b_0$ and $\gcd(a(t), b(t)) = 1$ such that condition (4.16) is valid.

By setting $\mathbf{z}_i = a_i + \mathbf{i}b_i$, $i = 1,2$ into $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ and substituting $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ into the left part of (4.16), we obtain

$$\frac{\mathrm{Im}(\overline{\boldsymbol{\alpha}}\boldsymbol{\alpha}' + \overline{\boldsymbol{\beta}}\boldsymbol{\beta}')}{|\boldsymbol{\alpha}|^2 + |\boldsymbol{\beta}|^2} = \frac{(b_1 + b_2)t^2 - 2(b_1 a_2 + a_1 b_2)t - b_2|\mathbf{z}_1|^2 - b_1|\mathbf{z}_2|^2 + |\mathbf{c}|^2 b_2}{[(t - a_1)^2 + b_1^2 + |\mathbf{c}|^2][(t - a_2)^2 + b_2^2]}. \tag{6.13}$$

Using (6.13) condition (4.16) becomes

$$\frac{(b_1 + b_2)t^2 - 2(b_1 a_2 + a_1 b_2)t - b_2|\mathbf{z}_1|^2 - b_1|\mathbf{z}_2|^2 + |\mathbf{c}|^2 b_2}{[(t - a_1)^2 + b_1^2 + |\mathbf{c}|^2][(t - a_2)^2 + b_2^2]} = \frac{-b_0}{(t - a_0)^2 + b_0^2} \tag{6.14}$$

which is equivalent to

$$(b_1 + b_2)\frac{t^2 - 2\frac{b_1 a_2 + a_1 b_2}{b_1 + b_2}t - \frac{b_2|\mathbf{z}_1|^2 + b_1|\mathbf{z}_2|^2 - |\mathbf{c}|^2 b_2}{b_1 + b_2}}{[(t - a_1)^2 + b_1^2 + |\mathbf{c}|^2][(t - a_2)^2 + b_2^2]} = \frac{-b_0}{(t - a_0)^2 + b_0^2}. \tag{6.15}$$

The denominator of the left side is a real polynomial of degree 4 and of the right a polynomial of degree 2. Hence the numerator of the left is required to be of degree 2 and so $b_1 + b_2 \neq 0$. We have the following two cases: either

$$t^2 - 2\frac{b_1 a_2 + a_1 b_2}{b_1 + b_2}t - \frac{b_2|\mathbf{z}_1|^2 + b_1|\mathbf{z}_2|^2 - |\mathbf{c}|^2 b_2}{b_1 + b_2} = (t - a_1)^2 + b_1^2 + |\mathbf{c}|^2$$

and

$$\frac{b_1 + b_2}{(t - a_2)^2 + b_2^2} = \frac{-b_0}{(t - a_0)^2 + b_0^2}$$

or

$$t^2 - 2\frac{b_1 a_2 + a_1 b_2}{b_1 + b_2}t - \frac{b_2|\mathbf{z}_1|^2 + b_1|\mathbf{z}_2|^2 - |\mathbf{c}|^2 b_2}{b_1 + b_2} = (t - a_2)^2 + b_2^2$$

and
$$\frac{b_1+b_2}{(t-a_1)^2+b_1^2+|\mathbf{c}|^2}=\frac{-b_0}{(t-a_0)^2+b_0^2}.$$
Thus condition (6.15) holds if and only if either
$$\begin{aligned}\frac{b_1a_2+a_1b_2}{b_1+b_2} &= a_1,\\ -\frac{b_2|\mathbf{z}_1|^2+b_1|\mathbf{z}_2|^2-|\mathbf{c}|^2b_2}{b_1+b_2} &= b_1^2+|\mathbf{c}|^2+a_1^2,\\ b_0^2 &= b_2^2,\\ -b_0 &= b_1+b_2,\\ a_0 &= a_2\end{aligned}$$

or
$$\begin{aligned}\frac{b_1a_2+a_1b_2}{b_1+b_2} &= a_2,\\ -\frac{b_2|z_1|^2+b_1|z_2|^2-|\mathbf{c}|^2b_2}{b_1+b_2} &= b_2^2+a_2^2,\\ b_0^2 &= b_1^2+|\mathbf{c}|^2,\\ b_0 &= -(b_1+b_2),\\ a_0 &= a_1.\end{aligned}$$

Combining the third and fourth equation of the first system we obtain $(b_1+b_2)^2=b_2^2$ and so $b_1=0$ or $b_1=-2\,b_2$. By setting $b_1=0$ we get the solution $(a_0,b_0)=(a_2,-b_2)$. If $b_1=-2\,b_2$ we have that $b_2=0$ which is a contradiction, since $\mathcal{A}(t)$ is primitive. Now by the fourth and third equation of the second system we have
$$|\mathbf{c}|^2=b_2^2+2\,b_1\,b_2 \tag{6.16}$$
and from the first equation we get $a_1=a_2$. Substituting the last two relations into the second we obtain $b_1=0$ or $b_1=-2\,b_2$. For $b_1=0$ the second system has the solution $(a_0,b_0)=(a_2,-b_2)$ and by substituting $b_1=-2\,b_2$ into the second equation we obtain $3\,b_2^2+|\mathbf{c}|^2=0$, which is a contradiction. Hence, condition (6.15) holds if and only if $b_1=0$.

Now a PH curve generated by the quaternion polynomial (6.12) is of type $(2,0)$, if and only if
$$(b_1+b_2)t^2-2(b_1a_2+a_1b_2)t+b_2|\mathbf{z}_1|^2+b_1|\mathbf{z}_2|^2+|\mathbf{c}|^2b_2=0 \tag{6.17}$$

which is equivalent to the following system

$$\begin{aligned} b_1 &= -b_2 \\ b_1 a_2 + a_1 b_2 &= 0 \\ b_1|\mathbf{z}_2|^2 + b_2|\mathbf{z}_1|^2 + |\mathbf{c}|^2 b_2 &= 0 \end{aligned}$$

From the second equation we get $b_2 = 0$ or $a_1 = a_2$. But since $\mathcal{A}(t)$ is primitive, $b_2 \neq 0$ and hence we study only the case $a_1 = a_2$. Substituting the last relation and $b_1 = -b_2$ into the third equation of the system we obtain $\mathcal{A}(t)$ is a real polynomial $\mathcal{A}(t)$, which is a contradiction. ∎

The following example shows how we can construct an RRMF curve of type $(2,1)$ with non–primitive hodograph and the purpose of Example 6.2.2 is to show how we can identify such a curve by using Corollary 2.3.

Example 6.2.1 Choosing the values $\mathbf{c} = 1$, $\mathbf{z}_1 = 0$ and $\mathbf{z}_2 = 1 - \mathbf{i}$, we have

$$\begin{aligned} \mathcal{A}(t) &= \boldsymbol{\alpha}(t) + \mathbf{k}\boldsymbol{\beta}(t) \\ &= t\,(t - \mathbf{z}_2) + \mathbf{k}\,(t - \mathbf{z}_2) \\ &= t^2 - t + \mathbf{i}\,t + \mathbf{k}\,(t - 1 + \mathbf{i}) \end{aligned}$$

and $\mathcal{A}(t)$ defines a non–primitive hodograph of an RRMF curve of type $(2,1)$. Easily one can verify that

$$\frac{\mathrm{Im}(\overline{\boldsymbol{\alpha}}\boldsymbol{\alpha}' + \overline{\boldsymbol{\beta}}\boldsymbol{\beta}')}{|\boldsymbol{\alpha}|^2 + |\boldsymbol{\beta}|^2} = \frac{-t^2 - 1}{t^4 - 2\,t^3 + 3\,t^2 - 2\,t + 2} = \frac{-1}{(t-1)^2 + 1}$$

and the complex polynomial $\mathbf{w}(t)$ which satisfy condition (4.16) is

$$\mathbf{w}(t) = t - 1 + \mathbf{i}.$$

The resulting hodograph is

$$\mathbf{r}'(t) = (x'(t), y'(t), z'(t)) = [\,t^4 - 2\,t^3 + t^2 + 2\,t - 2, t\,(t^2 - 2\,t + 2), 0\,]$$

and its components define a curve with a non–primitive hodograph and satisfies (4.3) where

$$\sigma(t) = (t^2 + 1)(t^2 - 2\,t + 2)$$

The hodograph defines a planar curve, as can be verified by Propoition 4.1.

Example 6.2.2 Let

$$\mathcal{A}(t) = t^2 - t + 2\,\mathbf{i}\,(1-t) - 2\,\mathbf{j} + \mathbf{k}\,t$$

be a quaternion polynomial which generates a hodograph of a quintic curve. By using Corollary 2.3 yields that $\mathcal{A}(t)$ has a complex root. The polynomial can also be written as

$$\mathcal{A}(t) = t^2 - t + 2\,\mathbf{i}\,(1-t) + \mathbf{k}\,(t - 2\,\mathbf{i})$$

and by above expression we obtain that the common root of $\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)$ is $\mathbf{z}_2 = 2\,\mathbf{i}$. Moreover, we get that $\mathbf{z}_1 = 1$, i.e., $Im(\mathbf{z}_1) = 0$. So, the polynomial $\mathcal{A}(t)$ defines an RRMF curve of type $(2,1)$. Note that by Proposition 4.1 the curve is planar.

The following Proposition is motivated by the previous examples.

Proposition 6.4. *Suppose that the polynomial (6.12) admits a factorization of the form*

$$\mathcal{A}(t) = \mathcal{B}(t)(t - \mathbf{z}) \tag{6.18}$$

with $\mathbf{z} \in \mathbb{C}$ *and defines the hodograph of the PH quintic curve* $\mathbf{r}(t)$*. Let* $\hat{\mathbf{r}}(t)$ *be the cubic defined by the polynomial* $\mathcal{B}(t)$*. Then*

1. *If* $\hat{\mathbf{r}}(t)$ *is an RRMF curve then the quintic* $\mathbf{r}(t)$ *is an RRMF curve of type* $(2,1)$*.*

2. *The curve* $\mathbf{r}(t)$ *is an RRMF of type* $(2,1)$ *if and only if it is planar.*

Proof: 1. By (6.18) and Lemma 4.1, we get

$$[\,\mathcal{A}(t)\,] = [\,\mathcal{B}(t)\,] + [t - \mathbf{z}] \tag{6.19}$$

and if $\hat{\mathbf{r}}(t)$ is an RRMF cubic curve then $\hat{\mathbf{r}}(t)$ is planar and this implies [7] that ERF=RMF. Thus $[\,\mathcal{B}(t)\,] = 0$. Consequently, relation (6.18) implies

$$[\,\mathcal{A}(t)\,] = [t - \mathbf{z}] = \frac{-b_0}{(t-a_0)^2 + b_0^2}, \quad \text{for} \;\; \mathbf{z} = a_0 + \mathbf{i}b_0,$$

and so $\mathcal{A}(t)$ defines an RRMF $\mathbf{r}(t)$ of type $(2,1)$.

2. Suppose that $\mathbf{r}(t)$ is an RRMF of type $(2,1)$. Since the hodograph is non–primitive by (6.19) we have

$$\mathcal{A}(t) = t^2 - a_0 t + \mathbf{i}\, b_0 t + \mathbf{j}\, b_0 + \mathbf{k}\,(t - a_0),$$

where $\mathbf{z}_2 = t - a_0 + \mathbf{i} b_0$. It is easily verified that $\mathcal{A}(t)$ satisfies condition (4.24) and thus $\mathbf{r}(t)$ is planar. Conversely, if $\mathbf{r}(t)$ is planar then by [34, Prop 2] a parameterization of $\mathcal{A}(t)$ is given by

$$\begin{aligned}\mathcal{A}(t) = {} & t^2 + u_1 t - (u_1 + r) r + \\ & \mathbf{i}\,[v_1 t + (u_1 + r) v_1] + \mathbf{j}\,(p_1 t + v_1 q_1 - p_1 r) + \mathbf{k}\,(q_1 t - v_1 p_1 - q_1 r)\end{aligned}$$

with r, u_1, v_1, p_1, q_1 are the free real variables. Expressing $\mathcal{A}(t)$ in power form we prove that $\mathcal{A}(t)$ verify (2.3) and thus $\mathbf{r}(t)$ is an RRMF of type $(2,1)$. ∎

Remark 6.5. Combining (3) of Proposition 6.4 and (2) of Proposition 6.2 we straightway have that RRMF curves of type $(2,1)$ which are generated by cubics are only the planar ones.

Example 6.2.3 Let

$$\mathcal{A}(t) = t^2 + t + \mathbf{i}\, t + 2\,\mathbf{j}\, t - 2\,\mathbf{k}\,(t + 2)$$

be a quaternion polynomial which defines the hodograph (Fig 7.1)

$$\mathbf{r}'(t) = (t^4 - 2\,t^3 + t^2 + 2\,t - 2, t\,(t^2 - 2\,t + 2), 0)$$

of a quintic PH curve. By Corollary 2.3 and Proposition 6.4, the quintic is an RRMF of type $(2,1)$ with non–primitive hodograph i.e., which is generated by a cubic curve. Since $\mathcal{A}(t)$ has a complex root can be expressed as

$$\mathcal{A}(t) = (t + \mathbf{k})(t - 1 + \mathbf{i})$$

and we deduce that the hodograph of the cubic (Fig. 7.2) is

$$\hat{\mathbf{r}}'(t) = (t + \mathbf{k})\,\mathbf{i}(t + \mathbf{k})^* = (t^2 - 1, 2\,t, 0).$$

The family of the curves having the hodograph $\mathbf{r}'(t)$ are

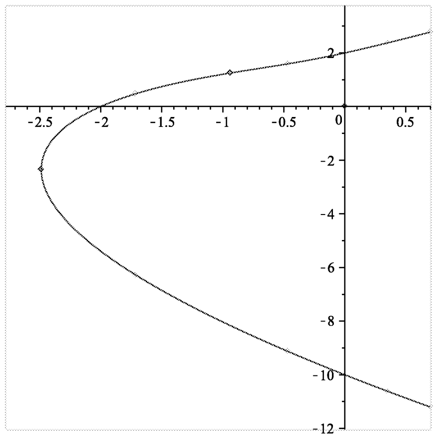

Figure 6.1 Curve $\mathbf{r}'(t)$.

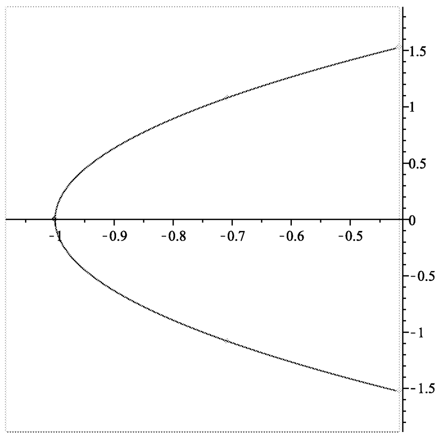

Figure 6.2 Curve $\hat{\mathbf{r}}'(t)$.

$$\mathbf{r}(t) = \left(\frac{t^5}{5} - \frac{t^4}{2} + \frac{t^3}{3} + t^2 - 2\,t + c_1,\ \frac{t^4}{2} - 4\frac{t^3}{3} - 2t^2 + c_2,\ c_3\right)$$

where c_1, c_2, c_3 are constant and for $c_1 = c_2 = c_3 = 0$ the graph is given in Fig. 6.3. Similarly, the family of the curves $\hat{\mathbf{r}}(t)$ are

$$\hat{\mathbf{r}}(t) = \left(\frac{t^3}{3} - t + c_1',\ t^2 + c_2',\ c_3'\right)$$

and for $c_1' = c_2' = c_3' = 0$ the graph is given in Fig. 6.4.

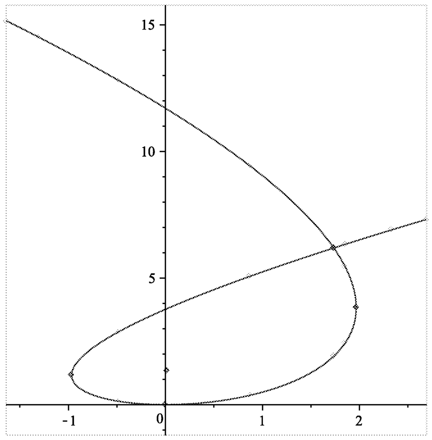

Figure 6.3 Curve $\mathbf{r}(t)$.

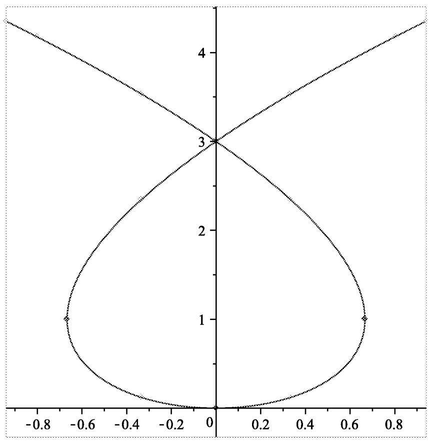

Figure 6.4 Curve $\hat{\mathbf{r}}(t)$.

6.2.2 Curves of type $(3,0)$ with non–primitive hodographs

In this section we study the set of degree 7 PH curves which they have a rotation–minimizing ERF and the polynomial $\mathcal{A}(t) \in \mathbb{H}[t]\setminus\mathbb{C}[t]$ is reducible. We shall prove that these curves are generated only by PH curves of type $(2,1)$.

Also, we shall give a parametrization of these curves in terms of the coefficients of type $(2,1)$ curves.

Characterization of RRMF curves of type $(3,0)$ with non–primitive hodographs

Our purpose here is first to find which set of PH curves of degree 5 generates the PH curves of type $(3,0)$. Then, we shall characterize/parametrize the set of the polynomials which defines the PH curves of type $(3,0)$.

Let $\mathcal{A}(t)$ be a cubic quaternion polynomial with a right complex factor

$$\mathcal{A}(t)=\mathcal{B}(t)(t-\mathbf{z}), \tag{6.20}$$

where $\mathcal{B}(t)\in\mathbb{H}[t]\setminus\mathbb{C}[t]$ and $\mathbf{z}\in\mathbb{C}\setminus\mathbb{R}$. The polynomial $\mathcal{A}(t)$ generates the hodograph $\mathbf{r}'(t)$ of a PH curve of degree 7 and $\mathcal{B}(t)$ generates the hodograph $\hat{\mathbf{r}}'(t)$ of the quintic one. We consider the following cases:

- 1st case: The polynomial $\mathcal{B}(t)$ is irreducible (level of non–primitivity 1).

Substituting

$$\mathcal{B}(t)=t^2+u_1't+u_0'+\mathbf{i}\,(v_1't+v_0')+\mathbf{j}\,(p_1't+p_0')+\mathbf{k}\,(q_1't+q_0')$$

and

$$\mathbf{z}=a+b\,\mathbf{i}$$

into (6.20), $\mathcal{A}(t)$ is expressed as

$$\begin{aligned}\mathcal{A}(t)\;=\;& t^3+(u_1'-a)\,t^2+(u_0'-au_1'-bv_1')\,t-au_0'-bv_0'\\&+\mathbf{i}\,[\,(v_1'+b)\,t^2+(v_0'-av_1'+bu_1')\,t+bu_0'-av_0'\,]\\&+\mathbf{j}\,[\,p_1't^2+(p_0'-ap_1'+bq_1')\,t+bq_0'-ap_0'\,]\\&+\mathbf{k}\,[\,q_1't^2+(q_0'-aq_1'-bp_1')\,t-aq_0'-bp_0'\,],\end{aligned} \tag{6.21}$$

If we consider that $\mathcal{A}(t)$ generates a PH curve of type $(3,0)$ then conditions (5.38) are satisfied by the polynomial $\mathcal{A}(t)$ and thus we have

$$\begin{aligned}v_1'+b&=0,\\v_0'-av_1'+bu_1'&=0,\\p_1'q_0'-p_1'^2b-p_0'q_1'-bq_1'^2&=3(bu_0'-av_0'),\end{aligned}$$

$$\begin{aligned} 0 \;=\; & bu_0'u_1' - au_1'v_0' - abu_0' + a^2v_0' + bq_0'q_1' \\ & -ap_0'q_1' + aq_0'p_1' + bp_0'p_1', \end{aligned}$$

and

$$\begin{aligned} 0 \;=\; & bu_0'^2 - au_0'v_0' - abu_0'u_1' + a^2u_1'v_0' \\ & -b^2u_0'v_1' + abv_0'v_1' + bq_0'^2 - b^2q_0'p_1' \\ & +a^2p_0'q_1' + bp_0'^2 - a^2q_0'p_1' + b^2p_0'q_1' \end{aligned}$$

The first two equations imply

$$b = -v_1' \quad \text{and} \quad a = \frac{v_0'}{v_1'} - u_1'$$

(with $b = -v_1' \neq 0$ since $\mathbf{z} \in \mathbb{C} \setminus \mathbb{R}$) and substituting to the next three we take the necessary and sufficient conditions for the coefficients of the $\hat{\mathbf{r}}'$ in order to generate an RRMF curve of type $(3,0)$:

$$p_1'q_0' - p_0'q_1' + v_1'(p_1'^2 + q_1'^2) = -3\left(v_1'u_0' + v_0'u_1' - \frac{v_0'^2}{v_1'}\right),$$

$$\begin{aligned} & -2u_0'u_1'v_1' + \frac{u_1'v_0'^2}{v_1'} + 2u_1'^2v_0' + u_0'v_0' + \frac{v_0'^3}{v_1'^2} \\ & \quad -v_1'(q_1'q_0' + p_1'p_0') - \left(\frac{v_0'}{v_1'} - u_1'\right)p_0'q_1' + (v_0' - u_1')q_0'p_1' = 0, \end{aligned}$$

$$\begin{aligned} & \left(\frac{v_0' - v_1'u_1'}{v_1'}\right)^2 (u_1'v_0' + p_0'q_1' - p_1'q_0') + (v_0' - v_1'u_1')\left(-\frac{u_0'v_0'}{v_1'} + u_1'u_0' - v_0'v_1'\right) + \\ & \quad v_1'^2(p_0'q_1' - q_0'p_1') + v_1'(p_0'^2 + q_0'^2) - u_0'v_1'(u_0' + v_1'^2) = 0. \end{aligned}$$

Remark 6.6. As we can observe the values of a and b depend only on the coefficients of $\mathcal{B}(t)$ and $\hat{\mathbf{r}}(t)$ generates an RRMF curve of type $(3,0)$, if the polynomial $\mathcal{B}(t)$ is multiplied by a specific complex factor of the form

$$t - \mathbf{z} = t - \frac{v_0'}{v_1'} + u_1' + v_1'\,\mathbf{i}.$$

Now we shall determine the set of curves $\mathbf{r}'(t)$.

Since $\mathcal{A}(t) = \mathcal{B}(t)(t-\mathbf{z})$, Lemma 4.1 implies that

$$[\mathcal{A}(t)] = [\mathcal{B}(t)(t-\mathbf{z})] = [\mathcal{B}(t)] + [t-\mathbf{z}]$$

and since $\mathbf{r}(t)$ is of type $(3,0)$, i.e. $[\mathcal{A}(t)] = 0$ then

$$[\mathcal{B}(t)] = -[t-\mathbf{z}] = \frac{b}{(t-a)^2+b^2}.$$

From the last relation, we see that $\hat{\mathbf{r}}'(t)$ is an RRMF curve of type $(2,1)$. Hence, we obtain that each RRMF curve of type $(3,0)$ with non–primitive hodograph is generated by an RRMF quintic of type $(2,1)$. The question that arises is if each RRMF of type $(2,1)$ generates a PH curve of type $(3,0)$. In fact, using [34, Prop. 2] we can see that relations (22) and (23) verify (6.22) and we deduce that each RRMF of type $(2,1)$ generates an RRMF curve of type $(3,0)$. Hence, by (6.21), [34, Prop. 2] and due to the fact that each PH curve of type $(3,0)$ is generated by a PH of type $(2,1)$ and vice versa, we may represent the set of RRMF curves of type $(3,0)$ -in terms of the coefficients of the PH of type $(2,1)$- as follows.

$$u_0 = (u_1' + a)(v_1'^2 + a^2), \qquad u_1 = -2u_1'a - a^2 + v_1'^2$$

$$v_0 = v_1 = v_2 = 0,$$

$$p_0 = p_1'(v_1'^2 + a^2), \quad p_1 = -2p_1'a,$$

$$q_0 = q_1'(v_1'^2 + a^2), \quad q_1 = -2q_1'a,$$

$$p_2 = p_1', \quad q_2 = q_1'. \tag{6.22}$$

or

$$
\begin{aligned}
u_0 &= (u_1'+a)(a^2+v_1'^2)+\frac{4av_1'^2(p_1'^2+q_1'^2)}{(u_1'+2a)^2+9v_1'^2+p_1'^2+q_1'^2}\\
u_1 &= -2u_1'a-a^2+v_1'^2-\frac{4v_1'^2(p_1'^2+q_1'^2)}{(u_1'+2a)^2+9v_1'^2+p_1'^2+q_1'^2}\\
u_2 &= u_1'-a\\
v_0 &= \frac{4v_1'^3(p_1'^2+q_1'^2)}{(u_1'+2a)^2+9v_1'^2+p_1'^2+q_1'^2}\\
v_1 &= 0\\
v_2 &= 0\\
p_0 &= p_1'(v_1'^2+a^2)-4v_1'^2\frac{v_1'[(u_1'+2a)q_1'+3v_1'p_1']+a[(u_1'+2a)p_1'-3v_1'q_1']}{(u_1'+2a)^2+9v_1'^2+p_1'^2+q_1'^2}\\
p_1 &= -2p_1'a+\frac{4v_1'^2[(u_1'+2a)p_1'-3v_1'q_1']}{(u_1'+2a)^2+9v_1'^2+p_1'^2+q_1'^2}\\
p_2 &= p_1'\\
q_0 &= q_1'(v_1'^2+a^2)+4v_1'^2\frac{v_1'[(u_1'+2a)p_1'-3v_1'q_1']-a[(u_1'+2a)q_1'+3v_1'p_1']}{(u_1'+2a)^2+9v_1'^2+p_1'^2+q_1'^2}\\
q_1 &= -2q_1'a+\frac{4v_1'^2[(u_1'+2a)q_1'+3v_1'p_1']}{(u_1'+2a)^2+9v_1'^2+p_1'^2+q_1'^2}\\
q_2 &= q_1'
\end{aligned}
\tag{6.23}
$$

where a, u_1', v_1', p_1', q_1' are free variables with $v_1'\neq 0$.

Note that relations (6.22) arise from the equations (22) of [34, Prop. 2] which represent the planar curves of type $(2,1)$ and equations (6.23) result from relations (23) which represent the true space curves of type $(2,1)$. We recall that so far we have proved that each PH curve of type $(3,0)$ with non–primitive hodograph is generated by a PH curve of type $(2,1)$ and conversely, each PH curve of type $(2,1)$ generates a PH curve of type $(3,0)$. Also he have characterized the coefficients of the PH curve of type $(3,0)$ with non–primitive hodograph in terms of the coefficients of the quintic which generates it, by given two different representation's types (6.22), (6.23) according to [34, Prop. 2].

In view of Proposition 6.2, the above results can be summarized as follows.

Proposition 6.5. *Let $\mathcal{A}(t)$ be a quaternion polynomial of the form (6.20) which defines the hodograph $\mathbf{r}'(t)$ of an RRMF curve of type $(3,0)$ and $\mathcal{B}(t)$*

an irreducible one which generates the hodograph $\hat{\mathbf{r}}'(t)$ *of an RRMF quintic curve. Then, the following hold*

1. *each curve* $\mathbf{r}(t)$ *is generated only by a curve* $\hat{\mathbf{r}}(t)$ *of type* $(2,1)$ *and conversely, each RRMF curve of type* $(2,1)$ *generates an RRMF of type* $(3,0)$*,*

2. *each planar of type* $(3,0)$ *is generated by planar of type* $(2,1)$ *and conversely,*

3. *each true spatial RRMF of type* $(3,0)$ *is generated by a true spatial RRMF of type* $(2,1)$ *and conversely.*

Moreover, the set of polynomials which defines the planar curves $\mathbf{r}(t)$ *is expressed by equations (6.22) - in terms of the coefficients of the* $\hat{\mathbf{r}}(t)$ *- and the set of true spatial curves is represented by equations (6.23).*

By using equations (6.22) and (6.23) we can generate planar or true space PH curves of type $(3,0)$, as the example below shows.

Example 6.2.4 Choosing the values $a = 1$, $u_1' = 2$, $v_1' = -1$, $p_1' = 0$, $q_1' = 1$ in (6.23) gives

$$u_0 = \frac{80}{13}, \quad u_1 = -\frac{54}{13}, \quad u_2 = 1, \quad v_0 = -\frac{2}{13}, \quad p_0 = \frac{2}{13},$$
$$p_1 = \frac{6}{13}, \quad p_2 = 0, \quad q_0 = \frac{12}{13}, \quad q_1 = -\frac{16}{13}, \quad q_2 = 1,$$

and hence we have

$$u(t) = t^3+t^2-\frac{54}{13}t+\frac{80}{13}, \quad v(t) = -\frac{2}{13}, \quad p(t) = \frac{6}{13}t+\frac{2}{13}, \quad q(t) = t^2-\frac{16}{13}t+\frac{12}{13}$$

which satisfy (5.37). The resulting hodograph components are

$$\begin{aligned}
x'(t) &= t^6 + 2\,t^5 - \frac{108}{13}\,t^4 + \frac{84}{13}\,t^3 + \frac{4392}{169}\,t^2 - \frac{8280}{169}\,t + \frac{6256}{169}, \\
y'(t) &= 2\,t^5 - \frac{6}{13}\,t^4 - \frac{116}{13}\,t^3 + \frac{4120}{169}\,t^2 - \frac{50080}{2197}\,t + \frac{24976}{2197}, \\
z'(t) &= -\frac{12}{13}\,t^4 - \frac{16}{13}\,t^3 + \frac{604}{169}\,t^2 - \frac{9800}{2197}\,t - \frac{4064}{2197}.
\end{aligned}$$

and the hodograph define a true space curve, as can also be verified from the fact that condition (6.14) is not satisfied. Note that the real polynomials that define the quintic curve are given by relations (23) of [34, Proposition 2],

$$u'(t) = t^2 + 2t - \frac{41}{13}, \quad v'(t) = -t - 3, \quad p(t) = -\frac{7}{13}, \quad q(t) = t - \frac{5}{13}$$

and they do not satisfy (6.14), as we expected.

- 2nd case: The polynomial $\mathcal{B}(t)$ is reducible (level of non–primitivity 2).

Consider that the polynomial $\mathcal{B}(t) \in \mathbb{H}[t] \setminus \mathbb{C}[t]$ has a right complex factor and so $\mathcal{B}(t) = (t - \mathcal{Q})(t - \mathbf{w})$ with $\mathcal{Q} = s_0 + s_1\mathbf{i} + s_2\mathbf{j} + s_3\mathbf{k}$ and $\mathbf{z} = c + d\,\mathbf{i} \in \mathbb{C} \setminus \mathbb{R}$. Now relation (6.20) is expressed as

$$\mathcal{A} = (t - \mathcal{Q})(t - \mathbf{w})(t - \mathbf{z}),$$

and by Lemma 4.1 implies

$$[\mathcal{A}] = [t - \mathcal{Q}] + [t - \mathbf{w}] + [t - \mathbf{z}].$$

Since $[\mathcal{A}(t)] = 0$ by the above relation we get

$$\frac{s_1}{(t - s_0)^2 + s_1^2 + s_2^2 + s_3^2} = -\frac{d}{(t - c)^2 + d^2} - \frac{b}{(t - a)^2 + b^2}.$$

which is valid if and only if

$$-s_1 = d + b, \quad s_0 = c = a, \quad s_1^2 + s_2^2 + s_3^2 = d^2 = b^2. \tag{6.24}$$

From (6.24) we have the following cases:

If $d = b$ then (6.24) gives

$$d = b = 0, \quad s_1 = s_2 = s_3 = 0,$$

and $\mathcal{A}(t) \in \mathbb{R}[t]$ which is contradiction since $\mathcal{A}(t) \in \mathbb{H}[t] \setminus \mathbb{C}[t]$.

If $d = -b$ from (6.24) implies that

$$s_1 = 0, \quad \text{vect}(\mathcal{Q}) = Im(\mathbf{w}) = Im(\mathbf{z}), \quad \text{scal}(\mathcal{Q}) = Re(\mathbf{w}) = Re(\mathbf{z})$$

which is equivalent to

$$\mathcal{Q} \sim \mathbf{w} \sim \mathbf{z}.$$

But if $\mathbf{w} \sim \mathbf{z}$ then the polynomial $\mathcal{A}(t)$ includes a real factor, which is a contradiction since $\mathcal{A}(t)$ is primitive. Thus, this case is discarded.

Finally, we give some examples of RRMF curves of type $(3,0)$ with non–primitive hodographs.

Example 6.2.5 Consider the quaternion polynomial

$$\begin{aligned}\mathcal{A}(t) &= t^3 + (\sqrt{5}-2)\,t^2 + \frac{5-\sqrt{5}}{10}\,t + \sqrt{5} - 2 + \frac{5+\sqrt{5}}{10}\,\mathbf{i} \\ &\quad + \mathbf{k}\left[t^2 + \frac{3\sqrt{5}+5}{10} + \left(t^2 - t - \frac{3\sqrt{5}-5}{10}\right)\mathbf{i}\right]\end{aligned}$$

which generates a PH curve $\mathbf{r}(t)$ of degree 7. By Theorem 6.1 and by expressing $\mathcal{A}(t)$ in quaternion form we can see that conditions (5.38) are verified and so $\mathcal{A}(t)$ defines an RRMF curve of type $(3,0)$ with non–primitive hodograph. Moreover, since

$$v_0 = \frac{5+\sqrt{5}}{10} \neq 0$$

the curve is a true space curve.

Example 6.2.6 Let the quaternion polynomial

$$\begin{aligned}\mathcal{A}(t) &= \left(t - 2 + \sqrt{5} + \frac{5+2\sqrt{5}}{20}\,\mathbf{i} + \frac{25-3\sqrt{5}}{40}\,\mathbf{j} + \frac{5-\sqrt{5}}{8}\,\mathbf{k}\right) \\ &\quad \left(t + \frac{15-2\sqrt{5}}{20}\,\mathbf{i} + \frac{15+3\sqrt{5}}{40}\,\mathbf{j} + \frac{3+\sqrt{5}}{8}\,\mathbf{k}\right)(t - \mathbf{i}).\end{aligned}$$

We can easily see that $\mathcal{A}(t)$ satisfies conditions (5.43) and thus it generates a PH curve of degree 7 with rotation minimizing ERF. The components $x'(t)$, $y'(t)$, $z'(t)$ of $\mathbf{r}'(t)$ and parametric speed $\sigma(t)$ are

$$\begin{aligned}x'(t) &= t^6 + (2\sqrt{5}-4)\,t^5 + \frac{40-21\sqrt{5}}{5}\,t^4 + \frac{14\sqrt{5}-30}{5}\,t^3 \\ &\quad + \frac{78-39\sqrt{5}}{5}\,t^2 + \frac{4\sqrt{5}-10}{5}\,t + \frac{43-18\sqrt{5}}{5},\end{aligned}$$

$$y'(t) = 2\,t^5 + \frac{13\,\sqrt{5}-15}{5}\,t^4 + \frac{10-2\,\sqrt{5}}{5}\,t^3 + \frac{12\,\sqrt{5}-14}{5}\,t^2 - \frac{2\,\sqrt{5}}{5}\,t + \frac{1-\sqrt{5}}{5},$$

$$\begin{aligned} z'(t) &= -2\,t^5 + (6-2\,\sqrt{5})\,t^4 + \frac{14\,\sqrt{5}-30}{5}\,t^3 \\ &\quad + \frac{55-21\,\sqrt{5}}{5}\,t^2 + \frac{14\,\sqrt{5}-20}{5}\,t + \frac{25-11\,\sqrt{5}}{5}, \end{aligned}$$

$$\begin{aligned} \sigma(t) &= t^6 + (2\,\sqrt{5}-4)\,t^5 + \frac{60-21\,\sqrt{5}}{5}\,t^4 + (4\,\sqrt{5}-8)\,t^3 \\ &\quad + \frac{105-42\,\sqrt{5}}{5}\,t^2 + (2\,\sqrt{5}-4)\,t + \frac{50-21\,\sqrt{5}}{5}. \end{aligned}$$

The hodograph $\mathbf{r}'(t)$ which is generated by $\mathcal{A}(t)$ is non–primitive since $\mathcal{A}(t)$ has a right complex factor and the curve $\mathbf{r}(t)$ is generated by another curve $\hat{\mathbf{r}}(t)$ which its hodograph $\hat{\mathbf{r}}'(t)$ is defined by the quadratic quaternion polynomial

$$\begin{aligned} \mathcal{B}(t) &= \left(t - 2 + \sqrt{5} + \frac{5+2\sqrt{5}}{20}\,\mathbf{i} + \frac{25-3\sqrt{5}}{40}\,\mathbf{j} + \frac{5-\sqrt{5}}{8}\,\mathbf{k}\right) \\ &\quad \left(t + \frac{15-2\sqrt{5}}{20}\,\mathbf{i} + \frac{15+3\sqrt{5}}{40}\,\mathbf{j} + \frac{3+\sqrt{5}}{8}\,\mathbf{k}\right), \end{aligned}$$

One can verify that $[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t) \neq 0$, for each t so $\mathbf{r}(t)$ is a true space curve.

Chapter 7

Closure

A moving frame along a curve describes the orientation of a rigid body along its trajectory. Frames which characterized by the fact that one component coincides with the curve unit tangent are called adapted frames. Rotation minimizing frame (RMF) is an adapted frame of special interest because it executes the least possible frame rotation along the curve. We are interested for curves with rational rotation minimizing frames (RRMF curves) since rational representations are preferred in practice. The search for RRMF curves is restricted to Pythagorean–hodograph curves and represented by the quaternion and Hoph map forms. The Euler-Rodrigues frame (ERF) is a key step in identifying RRMF curves since it is rational adapted frame defined on any spatial PH curve. The ERF is not in general an RMF. The first true spatial RRMF curves for which the ERF is itself rotation–minimizing (ERF=RMF) are PH curves of degree 7.

In the present thesis we studied some certain types of RRMF curves of degree 5 and 7. More particularly, we presented the sufficient and necessary conditions for PH curves– in terms of the coefficients of their associated quaternion polynomials– under which the curve is of type $(2, 2)$, $(2, 1)$ and $(2, 0)$. We also characterized the degree 7 PH curves with rotation–minimizing ERFs, i.e., PH curves of type $(3, 0)$ using both the quaternion and Hoph map forms. By using the Hoph map form the characterization was determined in terms of one real and two complex constraints on the curve coefficients. The complete categorization and characterization of all RRMF curves of degree 5 and 7 is a topic that deserves further research.

Since the hodograph of a PH curve is defined through a quaternion polynomial, we focused on the quadratic quaternion polynomials which generate PH curves of degree 5. More precisely, an algorithm to determine the roots of such polynomials was developed, based on the scalar–vector representation of quaternions and used to analyze the PH curves of type $(2, 2)$. Furthermore, in this thesis we characterize the regular PH curves with non–primitive hodographs and we proved that such curves generated by quaternion polynomials with a right complex factor. This was the motivation of presenting the necessary and sufficient condition for a quaternion polynomial in order to have a complex root. Finally, we characterize the sets of types $(2, 1)$, $(2, 0)$ and $(3, 0)$ RRMF curves with non–primitive hodographs. The study of RRMF curves of type $(n, 0)$ with $n > 3$ with non–primitive hodographs could be a subject of future investigation.

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